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THE FLOW PAST A THIN WING WITH AN OSCILLATING JET FLAP

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CONTENTS

An exact solution is obtained for the linearized flow past a thin two-dimensional wing of chord c at zero incidence in an incompressible stream of density ρ and undisturbed velocity U, with a thin jet of momentum-flux $2\rho U^2 c\mu$ emerging from its trailing edge at an oscillating deflexion-angle $\tau \exp(\mathrm{i}\omega Ut/c)$. The motion is governed by a singular third-order integro-differential equation, which becomes tractable when μ is small: solutions in this 'weak-jet' limit depend on a single parameter $\nu = \mu\omega$, and are found to exist only when $\nu \leqslant 2$. The possible significance of this critical frequency is discussed. Computations of jet shape and lift force for a range of values of ν are presented, and the solutions for periodic plunging and pitching motions of the wing are derived from that for deflexion. The formulation follows that of an earlier paper (Spence 1961 *b*), in which, however, an unsound approximation was made to the governing equations.

1. Introduction

The name 'jet flap' is given to a means of obtaining increased lift on a wing, ideally without loss of thrust, by allowing high velocity air from the jet engines to emerge in a narrow sheet from the trailing edge of the wing at some downward inclination to the undisturbed stream.

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The first experimental investigation of its properties was described by Schubauer (1930), in an M.Sc. thesis at the California Institute of Technology, but this remained unpublished. The principle was rediscovered by Hagedorn & Ruden (1938), and again after the advent of jet engines had made it a practical possibility, by independent teams of workers in France (Poisson-Quinton & Jousserandot 1955) and in England (Davidson 1956; Dimmock 1957). Subsequent work has been carried on in these and other countries, that up to 1960 having been reviewed by Williams, Butler & Wood (1960) at the second I.C.A.S. conference in Geneva. An analogy between the jet and a mechanical flap which explained many of the observed features was developed by Stratford (1956), and soon after this Woods (1958) in a more theoretical treatment exploited the similarity between the jet and the vortex wake behind an aerofoil in unsteady motion.

A more self-contained approach was proposed independently at about the same time by Helmbold (1955) in America, by Malavard (1956) and Legendre (1956) in France, and by the present author (Spence 1956) in England. These authors, like Woods, use the methods of thin-aerofoil theory, replacing wing and jet by vortex sheets, but by treating the jet sheet as infinitely thin (and therefore as containing fluid moving with infinite velocity, at a finite momentum-flux) they are able to obtain a simple boundary condition, namely that the pressure difference is proportional to the curvature along the jet, which is semi-infinite in extent. A second boundary condition of a different character is supplied by the requirement of no flow normal to the wing, so the problem is a mixed boundary-value one. It is very similar to others of this type arising in elasticity and in diffraction theory. Fairly complete numerical solutions were obtained by Spence (1956), and in a later paper (Spence 1961 *a*) the problem was explored further by analytical techniques. In particular it was found that for small values of a certain jet strength parameter μ (defined by equation (7) below) the integro-differential equation for the downwash distribution over the jet could be put into a form that is independent of μ, and admits a closed solution.

At about the same time Sears (private communication) called attention to the existence of a class of unsteady-flow problems involving the jet flap. Time-dependent motions could be produced in practice by control movements, by flight through gusty air, and by aeroelastic disturbances. The jet-flapped wing is so different from an ordinary wing, (the effective chord along which disturbances are propagated being of infinite extent), that one might expect qualitatively-different responses to time-dependent inputs from those of classical aerofoil theory.

Erickson (1962) has shown that provided the velocity of the jet normal to its boundaries is small compared with the streamwise velocity within it, the relation between pressure difference and curvature holds exactly as in steady flow. This is equation (5) below. It might be a poor approximation for a very high-frequency oscillation, but we shall continue to treat the jet velocity as infinite, in which case it holds for all frequencies, if only to obtain a consistent theory from which the main effects of time dependence might emerge. A preliminary investigation for unsteady flow (Spence 1961 *b*) was carried out by means of this formulation, and in particular the response of the jet shape to a step-wise change in its initial deflexion angle was considered. The investigation was made for the 'small μ' limit, and the main question discussed was the motion at small times after the change in deflexion. The treatment then involved a further approximation, namely the omission of space derivatives from the equations in comparison with time derivatives, after which a 'similarity' solution in terms of a variable $x/t^{\frac{2}{3}}$ existed. Subsequently Erickson has thrown doubt on this latter approximation, and in the hope of resolving this the problem has been re-examined using the full 'small μ' equations, i.e. those with both space and time derivatives present. What is presented now is therefore the first exact solution of the problem as it was posed in the paper cited.

The case considered is that of a steady oscillation in deflexion angle with reduced frequency ω. The 'small μ' equations treated are (24) and (25) below: these are precisely those quoted for the same case in Spence (1961 *b*), but wrongly simplified in that paper by omission of the derivatives on the left-hand sides. The equations depend not on ω by itself, but on the parameter $\nu = \mu\omega$, which must be retained even though μ is small in order to preserve physically-correct behaviour far from the wing. ν is effectively a Strouhal number. An unexpected finding was that the limiting equations possess a solution only for $\nu \leqslant 2$, so there is an upper limit $2/\mu$ on the frequency ω for which a solution is possible. This appears to be the underlying mathematical reason for the failure reported by Erickson of the earlier approximate solution for a transient motion, since this would be made-up in general of Fourier components of all frequencies. This emergence of a critical frequency, and the apparent inadequacy of the equations to treat sudden changes, raise interesting physical questions not only for the jet flap but for wake flows, which are closely related mathematically although containing a momentum defect in contrast to the momentum excess in a jet.

Outline of contents

The governing equations for a general unsteady motion, derived from the results of Spence (1961 *a*, *b*), are set out in § 2, and specialized in § 2·1 to the case indicated in figure 1 *a* of oscillatory flap-deflexion for a wing at zero incidence; the 'small μ' approximation is then introduced in § 2·2 to obtain a single integro-differential equation (22) for the jet ordinate h. This equation is solved in § 3, in three stages: first, a Laplace transformation is applied in § 3·1 to eliminate x derivatives; secondly, the resulting singular integral equation is solved in § 3·2 by methods due to Carleman (1922) and Muskhelishvili (1946) for the Laplace transform $\bar{h}$; and thirdly, this solution is inverted in §§ 3·3 and 3·4 to give the physical ordinate h. The limiting behaviour of the solution for small and large values of x is discussed in § 4·1, and an expression for the lift coefficient deduced from this in § 4·2. In § 4·3 we discuss the way in which the solution when ν is small approaches that found for steady flow ($\nu = 0$) by Spence (1961 *a*). In § 4·4 we sketch the way in which the approximation of § 2·2 can be improved by retaining extra terms in μ. A solution of the same form is still possible, but depends on both μ and ν, and the relation between ω and μ at the critical boundary becomes more complicated.

Some numerical computations of jet shape are described in § 5, and the remaining §§ 6 and 7 treat in outline the cases indicated in figures 1 *b* and 1 *c* in which the wing executes plunging and pitching motions respectively, while the jet emerges tangentially from the trailing edge. The solutions for these cases can be expressed in terms of that for the deflexion case. A general motion could be built up by the superposition of all three.

2. Governing equations

Equations derived more fully in Spence (1961 *a*, *b*) are summarized below. The system is governed by the singular integral equation

$$w(x) = -\frac{1}{2\pi}\int_0^\infty \frac{\gamma(x_1)\,\mathrm{d}x_1}{x_1-x}, \tag{1}$$

expressing the downward velocity on the axis in terms of the strength of a vortex distribution $\gamma(x)$ equal to the difference in streamwise velocity components immediately above and immediately below the wing and jet, which occupy the intervals $0<x<c$ and $x>c$ respectively. If we write

$$w(x) = \begin{cases} w_W(x) & (0 \leqslant x \leqslant c) \\ w_J(x) & (c<x<\infty), \end{cases} \tag{2}$$

together with a similar decomposition of $\gamma(x)$, then a standard inversion of (1) permits the elimination of $\gamma_W(x)$, leading to the equation

$$w_J(x) + \frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{\frac{1}{2}}\int_c^\infty \left(\frac{x_1}{x_1-c}\right)^{\frac{1}{2}}\frac{\gamma_J(x_1)\,\mathrm{d}x_1}{x_1-x} = -\frac{1}{\pi}\left(\frac{x-c}{x}\right)^{\frac{1}{2}}\int_0^c \left(\frac{x_2}{c-x_2}\right)^{\frac{1}{2}}\frac{w_W(x_2)\,\mathrm{d}x_2}{x_2-x}, \tag{3}$$

on the semi-infinite interval $x>c$. $w_W(x)$ is regarded here as known from the boundary condition on the wing (and will be taken as zero in the next sections).

The motion of the jet provides the kinematic equation

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)h_J = w_J, \tag{4}$$

U being the velocity of the undisturbed stream, and h_J the depth of the jet below a horizontal (streamwise) axis. In addition, the jump Δp say in pressure across the jet is related to its curvature κ by the equation

$$\Delta p = \kappa J, \tag{5}$$

where J is the momentum flux in the jet. When κ is replaced by $-\partial^2 h_J/\partial x^2$ and use is made of the unsteady form of Bernoulli's equation, this can be written

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\gamma_J = -2\mu c U^2\frac{\partial^3 h_J}{\partial x^3}, \tag{6}$$

where

$$\mu = \frac{J}{2\rho U^2 c} = \tfrac{1}{4}C_J. \tag{7}$$

Our object is to solve equations (3), (4) and (6) for a wing motion specified by $w_W(x,t)$ and a prescribed variation of $(\partial h_J/\partial x)\,(c,t)$, the jet slope at exit. From the solution we wish in particular to calculate the lift force, which is given in terms of the vorticity on the jet and the downwash on the wing by

$$L = \rho\left(U+\tfrac{1}{2}c\frac{\mathrm{d}}{\mathrm{d}t}\right)\Gamma_\infty(t) + \rho\frac{\partial}{\partial t}\left[-\int_c^\infty x^{\frac{1}{2}}(x-c)^{\frac{1}{2}}\gamma_J(x)\,\mathrm{d}x + 2\int_0^c x^{\frac{1}{2}}(c-x)^{\frac{1}{2}}w_W(x)\,\mathrm{d}x\right], \tag{8}$$

where $\Gamma_\infty(t)$ is the circulation round a circuit enclosing the wing and cutting the jet far downstream, and is expressible in terms of γ_J as

$$\Gamma_\infty(t) = \int_c^\infty \left(\frac{x}{x-c}\right)^{\frac{1}{2}}\gamma_J(x,t)\,\mathrm{d}x + 2\int_0^c\left(\frac{x}{c-x}\right)^{\frac{1}{2}}w_W(x)\,\mathrm{d}x \tag{9}$$

$$= 2\pi \lim_{x\to\infty} x w_J(x), \tag{10}$$

by (4). From (4) and (10) we infer that $w_J \sim \partial h_J/\partial t \sim \Gamma_\infty/2\pi x$ when x is sufficiently large, so that h_J then behaves like $1/x$, in contrast to the situation in steady flow when it is $\partial h_J/\partial x$ that behaves in this way, and $h_J \sim \ln x$.

2·1. *Oscillatory deflexion angle, wing at zero incidence*

We shall consider first the 'deflexion' case in which the wing remains in a fixed position at zero incidence throughout, that is

$$h_w(x,t) = 0 = w_W(x,t), \tag{11}$$

and the motion is induced by varying the exit angle of the jet. Since solutions for different boundary conditions at the wing are linearly superposable, that to be found now can

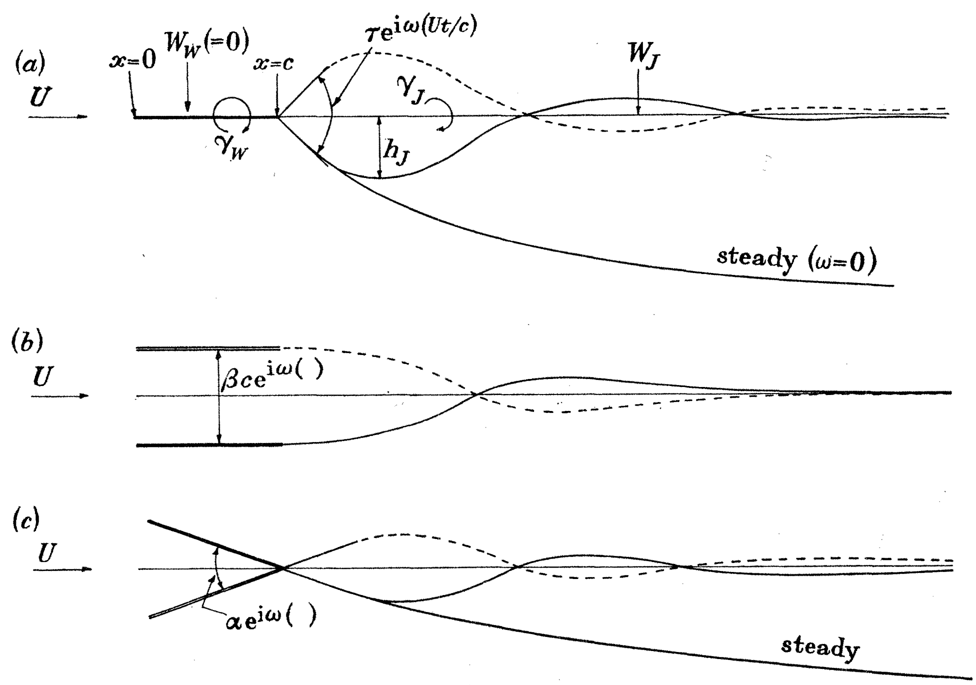

FIGURE 1. Modes of oscillation (schematic). (*a*) Deflexion-angle oscillating; (*b*) plunging motion; (*c*) pitching motion. The broken lines indicate the jet position 180° out of phase from the solid lines. Between these limits the phase advance is a function of position along the jet, and the points of zero deflexion vary during a cycle.

subsequently be combined with those for cases in which the wing moves but the jet emerges at all times tangentially at the trailing edge, as in the general pitching and plunging motion indicated in figures 1 (*b*) and 1 (*c*).

To treat an oscillation of reduced frequency ω we replace $h_J(x,t)$ and $\gamma_J(x,t)$ in the previous section by

$$e^{i\omega Ut/c} h_J(x), \quad e^{i\omega Ut/c} \gamma_J(x) \tag{12}$$

respectively, with boundary conditions

$$h'_J(c) = \tau \text{ (say)}, \quad h_J(c) = 0. \tag{13}$$

Equations (3) and (4) then combine to give

$$\left(i\omega + c\frac{d}{dx}\right) h_J(x) = -\frac{c}{2\pi U}\left(\frac{x-c}{x}\right)^{\frac{1}{2}} \int_c^\infty \left(\frac{x_1}{x_1-c}\right)^{\frac{1}{2}} \frac{\gamma_J(x_1)\,dx_1}{x_1-x}, \tag{14}$$

while (6) becomes
$$\left(\mathrm{i}\omega + c\frac{\mathrm{d}}{\mathrm{d}x}\right)\gamma_J(x) = -2\mu Uc^2\frac{\mathrm{d}^3h_J}{\mathrm{d}x^3}. \tag{15}$$

Our object is to solve this pair of equations, subject to the stated boundary conditions. The equations can be combined, as is shown in appendix A, into the single integro-differential equation

$$\left(\mathrm{i}\omega + c\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 h_J(x) - \frac{\mu c^3}{\pi}\int_c^\infty \left[\frac{x_1(x_1-c)}{x(x-c)}\right]^{\frac{1}{2}} \frac{h_J'''(x_1)}{x_1-x}\,\mathrm{d}x_1$$
$$= \frac{c}{2\pi U x^{\frac{1}{2}}(x-c)^{\frac{1}{2}}}\int_c^\infty \left(\mathrm{i}\omega - \frac{c^2}{2xx_1}\right)\left(\frac{x_1}{x_1-c}\right)^{\frac{1}{2}}\gamma_J(x_1)\,\mathrm{d}x_1. \tag{16}$$

An analytic solution of this equation as it stands does not seem possible, although a numerical solution of the type used by Spence (1956) for the corresponding equation for steady flow could almost certainly be found by approximating to h_J by means of a finite number of terms of a Fourier series in $\theta = 2\cos^{-1}(c/x)^{\frac{1}{2}}$. We shall however consider here the approximate form to which (16) reduces when the jet-strength parameter μ is small compared with unity (i.e. when C_J is small compared with 4, which is certainly the case in any practical system). As in the investigation for steady flow reported in Spence (1961 *a*), we 'stretch' the trailing edge region by use of the co-ordinate

$$\bar{x} = (x-c)/\mu c, \quad \text{i.e.} \quad x/c = 1+\mu\bar{x}. \tag{17}$$

With the substitution
$$\left.\begin{aligned} h_J(x) &= \mu c\tau(x/c)^{-\frac{1}{2}}h(\bar{x}),\\ \gamma_J(x) &= 2U\tau(x/c)^{-\frac{1}{2}}g(\bar{x}),\end{aligned}\right\} \tag{18}$$

(16) becomes

$$\left[\left(\mathrm{i}\nu + \frac{\mathrm{d}}{\mathrm{d}\bar{x}}\right)^2 - \left(\frac{\mu}{1+\mu\bar{x}}\right)\left(\mathrm{i}\nu + \frac{\mathrm{d}}{\mathrm{d}\bar{x}}\right) + \frac{3}{4}\left(\frac{\mu}{1+\mu\bar{x}}\right)^2\right]h(\bar{x})$$
$$-\frac{1}{\pi}\int_0^\infty \left(\frac{\bar{x}_1}{\bar{x}}\right)^{\frac{1}{2}}\left[h'''(\bar{x}_1) - \frac{3}{2}\left(\frac{\mu}{1+\mu\bar{x}_1}\right)h''(\bar{x}_1) + \frac{9}{4}\left(\quad\right)^2 h'(\bar{x}_1) - \frac{15}{8}\left(\quad\right)^3 h(\bar{x}_1)\right]\frac{\mathrm{d}\bar{x}_1}{\bar{x}_1-\bar{x}}$$
$$= \frac{1}{\pi\bar{x}^{\frac{1}{2}}}\int_0^\infty \left\{\mathrm{i}\nu - \frac{\frac{1}{2}\mu}{(1+\mu\bar{x})(1+\mu\bar{x}_1)}\right\}\bar{x}_1^{-\frac{1}{2}}g(\bar{x}_1)\,\mathrm{d}\bar{x}_1, \tag{19}$$

where
$$\nu = \mu\omega, \tag{20}$$

and from (13)
$$h'(0) = 1, \quad h(0) = 0. \tag{21}$$

2·2. *Approximation for small* μ

A first approximation for small μ, valid uniformly for $0 < x < \infty$, is obtained by setting $\mu = 0$ wherever it occurs explicitly in (19), but retaining the terms in ν which, although unimportant when x is small, are necessary to ensure that the asymptotic behaviour of the solution for large x shall be physically correct (for, since h behaves asymptotically like a negative power of x, $\nu h \gg \mathrm{d}h/\mathrm{d}x$ wherever $x \gg 1/\nu$).

The limiting equation to be solved is thus

$$\mathscr{L}h(\bar{x}) \equiv \left(\mathrm{i}\nu + \frac{\mathrm{d}}{\mathrm{d}\bar{x}}\right)^2 h(\bar{x}) - \frac{1}{\pi}\int_0^\infty \left(\frac{\bar{x}_1}{\bar{x}}\right)^{\frac{1}{2}}\frac{h'''(\bar{x}_1)\,\mathrm{d}\bar{x}_1}{\bar{x}_1-\bar{x}} = \frac{\mathrm{i}\nu A}{\bar{x}^{\frac{1}{2}}} \tag{22}$$

say, where
$$A = \mathrm{i}\nu\lim_{\bar{x}\to\infty}\bar{x}^{\frac{1}{2}}h(\bar{x}) = \frac{1}{\pi}\int_0^\infty \bar{x}_1^{-\frac{1}{2}}g(\bar{x}_1)\,\mathrm{d}\bar{x}_1 = \frac{\Gamma_\infty}{2\pi U\tau c\mu^{\frac{1}{2}}}, \tag{23}$$

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the boundary conditions being given by (21) as before. We have replaced the expression (9) by $\Gamma_\infty \exp(\mathrm{i}\omega Ut/c)$. At this stage its magnitude, and therefore that of A, might appear to be a disposable parameter, but it will be found that for each ν a solution of (22) exists for just one value $A(\nu)$, which is therefore more properly looked on as an eigenvalue of the system. In fact it is ultimately found (equation (79) below) that $|A(\nu)|$ has the constant value $\pi^{-\frac{1}{2}}$ for all ν.

Before proceeding to this solution the limiting forms of equations (14) and (15) following the substitution (18) may be noted. As $\mu \to 0$ these are

$$\left(\mathrm{i}\nu+\frac{\mathrm{d}}{\mathrm{d}\bar{x}}\right)h(\bar{x}) = -\frac{1}{\pi}\int_0^\infty \left(\frac{\bar{x}}{\bar{x}_1}\right)^{\frac{1}{2}} \frac{g(\bar{x}_1)\,\mathrm{d}\bar{x}_1}{\bar{x}_1-\bar{x}}, \tag{24}$$

$$\left(\mathrm{i}\nu+\frac{\mathrm{d}}{\mathrm{d}\bar{x}}\right)g(\bar{x}) = -h'''(\bar{x}). \tag{25}$$

These can be combined to yield (22) by the same method as used in appendix A for the full equations. They are precisely the equations used to treat oscillatory motion by Spence (1961 *b*), but further approximations were then made. These are discussed in the light of the present exact solution in § 8 below.

3. Solution of equation (22)

The solution can be obtained by a method similar to that used by Spence (1961 *a*) to treat the corresponding equation for steady flow (which is obtained by setting $\nu = 0$ in (22)). From now on the bar over x in (22) *et seq.* will be omitted. There are three stages in the solution: first, an integral equation is derived from (22) for the Laplace transform of $h(x)$, defined by

$$\bar{h}(\xi) = \xi\int_0^\infty \mathrm{e}^{-\xi x} h(x)\,\mathrm{d}x; \tag{26}$$

secondly, the equation is solved by the method of Carleman (1922); and thirdly, the resulting expression for $\bar{h}(\xi)$ is inverted to yield $h(x)$. The inversion can only be accomplished if $\nu < 2$ and then only for one value of $A(\nu)$, which is therefore fixed at that stage.

3·1. *Equation for* $\bar{h}(\xi)$

This is found by applying the Laplace operator to the terms of (22), but some preliminary manipulation of the integrand in that equation is necessary to allow for the behaviour of $h'''(x)$ near $x = 0$. This behaviour can be inferred from (24) and (25) without solving these equations in full. As $x \to 0$ the left-hand side of (24) tends to unity, by (21), and the fact that

$$\frac{1}{\pi}\int_0^\infty \left(\frac{x}{x_1}\right)^{\frac{1}{2}} \frac{(\ln x_1)\,\mathrm{d}x_1}{x_1-x} = \pi \quad (x > 0) \tag{27}$$

enables us to deduce that $g(x) \sim -(1/\pi)\ln x$ as $x \to 0$, and hence from (25) that

$$h'''(x) \sim 1/\pi x \quad \text{as} \quad x \to 0. \tag{28}$$

Therefore, since
$$\int_0^\infty \frac{\mathrm{d}x_1}{x_1^{\frac{1}{2}}(x_1-x)} = 0 \quad (x > 0), \tag{29}$$

equation (22) is unaffected if $h'''(x_1)$ is replaced in the integrand by

$$h'''(x_1)-1/\pi x_1, \quad \equiv \psi(x_1) \text{ say.} \tag{30}$$

The Laplace transform of the integral in (22) is then found, on writing $x_1 = xu$, as

$$\xi\int_0^\infty \mathrm{e}^{-\xi x}\,\mathrm{d}x\left\{\int_0^\infty \frac{u^{\frac{1}{2}}\psi(xu)\,\mathrm{d}u}{u-1}\right\},$$

and if we integrate first with respect to x and then with respect to u, this becomes

$$\int_0^\infty \frac{u^{\frac{1}{2}}\bar{\psi}(\xi/u)\,\mathrm{d}u}{u-1} = -\int_0^\infty\left(\frac{\xi}{\xi_1}\right)^{\frac{3}{2}}\frac{\bar{\psi}(\xi_1)\,\mathrm{d}\xi_1}{\xi_1-\xi}. \tag{31}$$

The transform of (30) is

$$\bar{\psi}(\xi_1) = \xi_1^3\bar{h}(\xi_1)-\xi_1^2+(\xi_1/\pi)\ln\xi_1+\xi_1\left[(\gamma/\pi)-\lim_{x\to 0}\{h''(x)-(1/\pi)\ln x\}\right], \tag{32}$$

where γ is Euler's constant. Insertion of this result in (31) and use of (29) enables us to write the Laplace transform of (22) as

$$\left(1+\frac{\mathrm{i}\nu}{\xi}\right)^2\bar{h}(\xi)+\frac{1}{\pi}\int_0^\infty\left(\frac{\xi_1}{\xi}\right)^{\frac{1}{2}}\frac{\xi_1\bar{h}(\xi_1)-1}{\xi_1-\xi}\,\mathrm{d}\xi_1 = \frac{\mathrm{i}\nu\pi^{\frac{1}{2}}}{\xi^{\frac{3}{2}}}A. \tag{33}$$

3·2. *Solution of equation* (33)

Consider the function

$$F(\zeta) = \frac{1}{\pi}\int_0^\infty\left(\frac{\xi_1}{\zeta}\right)^{\frac{1}{2}}\frac{\xi_1\bar{h}(\xi_1)-1}{\xi_1-\zeta}\,\mathrm{d}\xi_1 \quad (0<\arg\zeta<2\pi), \tag{34}$$

which is analytic in the whole ζ-plane cut along the positive real axis. Its values at points $\xi\pm\mathrm{i}0$ $(\xi>0)$ immediately above and immediately below the axis are connected by the Plemelj formulae

$$\left.\begin{aligned} F^+(\xi)+F^-(\xi) &= 2\mathrm{i}\{\xi\bar{h}(\xi)-1\},\\ F^+(\xi)-F^-(\xi) &= \frac{2}{\pi}\int_0^\infty\left(\frac{\xi_1}{\xi}\right)^{\frac{1}{2}}\frac{\xi_1\bar{h}(\xi_1)-1}{\xi_1-\xi},\end{aligned}\right\} \tag{35}$$

(The square root under the integral sign in (34) causes the formulae for the sum and difference to be the reverse of those usually quoted.)

Define also

$$\theta(\xi) = \frac{1}{2\mathrm{i}}\ln\left[\frac{\xi^3+\mathrm{i}(\xi+\mathrm{i}\nu)^2}{\xi^3-\mathrm{i}(\xi+\mathrm{i}\nu)^2}\right], \tag{36}$$

the branch of the logarithm being chosen so that θ decreases from π to 0 as ξ describes the real axis from $-\infty$ to ∞.

On substitution of (35) and (36) in (33) we replace the integral equation by a Hilbert boundary value problem

$$F^+\mathrm{e}^{-\mathrm{i}\theta}-F^-\mathrm{e}^{\mathrm{i}\theta} = -2\sin\theta+2\mathrm{i}\nu\pi^{\frac{1}{2}}A\xi^{-\frac{3}{2}}\cos\theta. \tag{37}$$

To solve for $F(\zeta)$, consider the function

$$X(\zeta) = \frac{1}{\pi}\int_0^\infty\frac{\theta(\xi_1)\,\mathrm{d}\xi_1}{\xi_1-\zeta}, \tag{38}$$

which is analytic for $0 < \arg\zeta < 2\pi$. As $\zeta \to \xi \pm i0$ $(\xi > 0)$ the values of $X(\zeta)$ are

$$X^{\pm}(\xi) = \Omega(\xi) \pm i\theta(\xi), \tag{39}$$

say, where

$$\Omega(\xi) = \frac{1}{\pi}\int_0^\infty \frac{\theta(\xi_1)\,d\xi_1}{\xi_1 - \xi}, \tag{40}$$

the integral being the Cauchy principal value, defined only for real ξ. Both Ω and θ are in general complex.

Division of (37) by $\exp\Omega(\xi)$ gives

$$(F\,e^{-X})^+ - (F\,e^{-X})^- = -2(\sin\theta - i\nu A\pi^{\frac{1}{2}}\xi^{-\frac{3}{2}}\cos\theta)\,e^{-\Omega}. \tag{41}$$

This can be looked on as one of the Plemelj formulae (defined in the more usual way, without the square root that occurs in (35)), for the function $F(\zeta)\,e^{-X(\zeta)}$. The complementary formula is then

$$(F\,e^{-X})^+ + (F\,e^{-X})^- = \frac{2i}{\pi}\int_0^\infty e^{-\Omega(\xi_1)}\,[\sin\theta(\xi_1) - i\nu A\pi^{\frac{1}{2}}\xi_1^{-\frac{3}{2}}\cos\theta(\xi_1)]\,\frac{d\xi_1}{\xi_1 - \xi}. \tag{42}$$

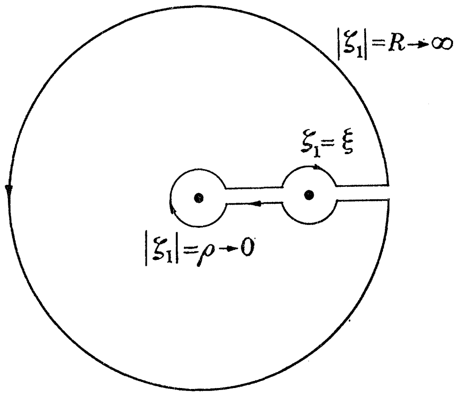

FIGURE 2. Contour for evaluating (43).

(The possibility of an arbitrary polynomial appearing on the right-hand side of (42) is excluded by the fact that by its definition $F \to 0$ as $|\zeta| \to \infty$.) The part of the integral containing $\sin\theta(\xi_1)$ in (42) can be found by evaluating

$$\frac{1}{2\pi i}\oint \frac{e^{-X(\zeta_1)}}{\zeta_1 - \xi}\,d\zeta_1 \tag{43}$$

round a contour consisting of the infinite circle together with the two sides of the positive real axis with semicircular indents above and below the point $\zeta_1 = \xi$, and a vanishingly small circle with centre the origin (figure 2).

Since, as $|\zeta| \to \infty$,

$$X(\zeta) = -\frac{\ln|\zeta|}{\pi\zeta} + O\,|\zeta|^{-1} \tag{44}$$

(this result follows from the fact that $\theta(\xi) = \xi^{-1} + O(\xi^{-2})$ as $\xi \to \infty$), the contribution from the infinite circle is unity, and we finally obtain

$$\frac{1}{\pi}\int_0^\infty \frac{e^{-\Omega(\xi_1)}\sin\theta(\xi_1)\,d\xi_1}{\xi_1 - \xi} = 1 - e^{-\Omega(\xi)}\cos\theta(\xi). \tag{45}$$

Likewise evaluation of

$$\frac{1}{2\pi i}\oint \frac{e^{-X(\zeta_1)}}{\zeta_1^{\frac{3}{2}}(\zeta_1 - \xi)} \tag{46}$$

round the same contour gives

$$\frac{1}{\pi}\int_0^\infty \frac{e^{-\Omega(\xi_1)}\cos\theta(\xi_1)\,d\xi_1}{\xi_1^{\frac{3}{2}}(\xi_1-\xi)} = -\xi^{-1}e^{-k_0}+\xi^{-\frac{3}{2}}e^{-\Omega(\xi)}\sin\theta(\xi), \tag{47}$$

where k_0 is defined by the expansion of $X(\zeta)$ near $\zeta = 0$, which can be written

$$\left.\begin{aligned} X(\zeta) &= -\tfrac{1}{2}\ln\zeta+\tfrac{1}{2}i\pi+k_0+\zeta k_1+\tfrac{1}{2}\zeta^2k_2+O(\zeta^3\ln\zeta),\\ \text{where}\quad k_0(\nu) &= -\frac{1}{\pi}\int_0^\infty \theta'(\xi)\ln\xi\,d\xi.\\ k_n(\nu) &= \frac{1}{\pi}\int_0^\infty \theta'(\xi)\,d\xi/\xi^n \quad (n=1,2). \end{aligned}\right\} \tag{48}$$

On solving (41) and (42) after insertion of these results we find

$$\left.\begin{aligned} F^{\pm} &= i(1+A'\xi^{-1})\,e^{\Omega\pm i\theta}-(i\mp A'e^{k_0}\xi^{-\frac{3}{2}}),\\ \text{where}\quad A' &= i\nu A\pi^{\frac{1}{2}}e^{-k_0}. \end{aligned}\right\} \tag{49}$$

Substitution in (35) then gives

$$\bar{h}(\xi) = \frac{\xi(\xi+A')}{\sqrt{\{\xi^6+(\xi+i\nu)^4\}}}\,e^{\Omega(\xi)}. \tag{50}$$

This form of the solution is not suitable for Laplace inversion as it stands since $\Omega(\xi)$ is defined only on the positive real axis. We now need an analytic continuation of (50) into the whole ζ-plane.

3·3. *Analytic continuation of* $\bar{h}(\xi)$

To find this continuation, define the function

$$\begin{aligned} Z(\zeta) &= \frac{1}{\pi}\int_{-\infty}^0 \frac{\theta(\xi_1)-\pi}{\xi_1-\zeta}\,d\xi_1 \quad (-\pi<\arg\zeta<\pi)\\ &= -\tfrac{1}{2}\ln\zeta-\frac{1}{\pi}\int_{-\infty}^0 \theta'(\xi_1)\ln(\zeta-\xi_1)\,d\xi_1 \end{aligned} \tag{51}$$

on integration by parts. Adding this to (38), similarly integrated by parts, we obtain

$$X(\zeta)+Z(\zeta) = -\ln\zeta-\frac{1}{\pi}\int_{-\infty}^\infty \theta'(\xi_1)\ln(\zeta-\xi_1)\,d\xi_1. \tag{52}$$

If $\mathscr{I}\,\zeta<0$ the integral on the right can be found by evaluating

$$\frac{1}{\pi}\oint \theta'(\zeta_1)\ln(\zeta-\zeta_1)\,d\zeta_1 \tag{53}$$

round the infinite semicircle on the upper side of the real axis in the ζ_1 plane (figure 3). The contribution from the curved part is vanishingly small as the radius tends to infinity, so the integral along the real axis is equal to the sum of residues of $2i\theta'(\zeta_1)\ln(\zeta-\zeta_1)$ at poles inside the contour. To find these poles, factorize the cubics in (36) in the form

$$\left.\begin{aligned} \zeta^3-i(\zeta+i\nu)^2 &= \prod_{i=1}^3(\zeta-\alpha_i) = \Pi^+(\zeta-\alpha_i)\,\Pi^-(\zeta-\alpha_i),\\ \zeta^3+i(\zeta+i\nu)^2 &= \prod_{i=1}^3(\zeta-\beta_i) = \Pi^+(\zeta-\beta_i)\,\Pi^-(\zeta-\beta_i), \end{aligned}\right\} \tag{54}$$

say, where $\Pi^+(\zeta-\alpha_i)$ is the product of factors for which $\mathscr{I}\alpha_i > 0$, $\Pi^-(\zeta-\alpha_i)$ is the product of those for which $\mathscr{I}\alpha_i < 0$, and the $\Pi^\pm(\zeta-\beta_i)$ are similarly defined. Differentiation of (36) gives

$$\theta'(\zeta_1) = \frac{1}{2\mathrm{i}}\sum_{i=1}^{3}\left(\frac{1}{\zeta_1-\beta_1}-\frac{1}{\zeta_1-\alpha_i}\right), \tag{55}$$

so the integral (53) receives contributions $+\ln(\zeta-\beta_i)$, $-\ln(\zeta-\alpha_i)$ from poles β_i, α_i in the upper-half plane. Summing such contributions we see that

$$\underline{\mathscr{I}\zeta < 0:}\quad \frac{1}{\pi}\int_{-\infty}^{\infty}\theta'(\xi_1)\ln(\zeta-\xi_1)\,\mathrm{d}\xi_1 = \ln\frac{\Pi^+(\zeta-\beta_i)}{\Pi^+(\zeta-\alpha_i)}. \tag{56}$$

Likewise integrating round the lower semicircle gives the value

$$-\ln\frac{\Pi^-(\zeta-\beta_i)}{\Pi^-(\zeta-\alpha_i)} \tag{57}$$

for the same integral when $\underline{\mathscr{I}\zeta > 0}$.

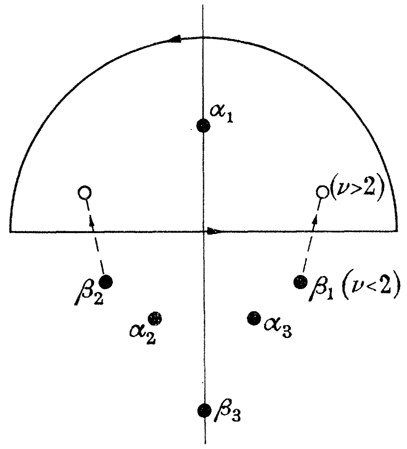

FIGURE 3. Contour for evaluating (53).

Now use superscripts +, − to denote values at $\zeta = \xi \pm \mathrm{i}0$, where ξ is now free to range from $-\infty$ to ∞. Then from (52), (56) and (57)

$$X^+(\xi)+Z^+(\xi) = \ln\frac{\Pi^-(\xi-\beta_i)}{\Pi^-(\xi-\alpha_i)}-\ln\xi, \tag{58}$$

$$X^-(\xi)+Z^-(\xi) = -\ln\frac{\Pi^+(\xi-\beta_i)}{\Pi^+(\xi-\alpha_i)}-\ln\xi. \tag{59}$$

If $\xi > 0$, $Z^+(\xi) = Z^-(\xi) = Z(\xi)$, and adding (58) and (59) with use of (39) gives

$$2\{Z(\xi)+\Omega(\xi)\} = \ln\prod_{i=1}^{3}(\xi-\alpha_i)(\xi-\beta_i) - 2\ln[\Pi^-(\xi-\alpha_i)\,\Pi^+(\xi-\beta_i)] - 2\ln\xi,$$

or

$$\frac{\mathrm{e}^{\Omega(\xi)}}{\sqrt{(\xi^6+(\xi+\mathrm{i}\nu)^4)}} = \frac{\mathrm{e}^{-Z(\xi)}}{\xi\Pi^-(\xi-\alpha_i)\,\Pi^+(\xi+\beta_i)}. \tag{60}$$

Substitution of the expression on the left into (50) then gives

$$\bar{h}(\zeta) = \frac{(\zeta+A')\,\mathrm{e}^{-Z(\zeta)}}{\Pi^-(\zeta-\alpha_i)\,\Pi^+(\zeta-\beta_i)}. \tag{61}$$

This is the required continuation, and it remains to locate the roots of the cubics. For small positive ν these are

$$\left.\begin{matrix}\alpha_1\\ \beta_1\end{matrix}\right\} = \pm\mathrm{i}(1\pm 2\nu-3\nu^2), \quad \left.\begin{matrix}\alpha_2\\ \alpha_3\end{matrix}\right\} = -\mathrm{i}\nu(1\mp\mathrm{i}\nu^{\frac{1}{2}}-\tfrac{3}{2}\nu), \quad \left.\begin{matrix}\beta_2\\ \beta_3\end{matrix}\right\} = -\mathrm{i}\nu(1\pm\nu^{\frac{1}{2}}+\tfrac{3}{2}\nu) \tag{62}$$

up to order ν^2, and more generally an examination in appendix B shows that for $\nu > 0$:

(i) Of the α_i, one, namely α_1, lies on the positive imaginary axis, while α_2 and α_3 are in the third and fourth quadrants respectively, at image points in the negative imaginary axis.

(ii) Of the β_i, β_3 always lies on the negative imaginary axis.

If $0 < \nu < \frac{4}{27}$, β_1 and β_2 also lie on the negative imaginary axis (as in (62)). If $\frac{4}{27} < \nu < 2$, β_1 and β_2 are in the negative half plane, at image points in the imaginary axis. If $\nu > 2$, β_1 and β_2 are in the positive half plane, at image points in the imaginary axis. (For $\nu = 2$ the roots are ± 2, $-\mathrm{i}$.) Thus if $\nu < 2$ the denominator of (61) is just

$$(\zeta-\alpha_2)(\zeta-\alpha_3), \tag{63}$$

whereas if $\nu > 2$ it contains the further factors $(\zeta-\beta_1)(\zeta-\beta_2)$. In the special case $\nu = 2$ the arguments leading to equation (61) require revision on account of the singularities on the real axis, but this will not be considered here.

3·4. *Inversion of* $\bar{h}(\xi)$

The standard formula gives

$$h(x) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \bar{h}(\zeta)\,\mathrm{e}^{\zeta x}\,\mathrm{d}\zeta/\zeta, \tag{64}$$

where $\mathscr{R}\zeta = c$ lies to the right of all singularities of the integrand, namely the branch point at $\zeta = 0$ (where $\bar{h}(\zeta)$ behaves like $\zeta^{\frac{1}{2}}$) and, for $\nu < 2$ and general values of A', the poles at $\zeta = \alpha_2$ and $\zeta = \alpha_3$. When the path of integration is displaced to the left, the integral receives contributions from these latter proportional to $\mathrm{e}^{\alpha_2 x}$ and $\mathrm{e}^{\alpha_3 x}$ respectively. However since α_3 has a positive real part its contribution would make $h(x)$ exponentially large as $x\to\infty$, thereby rendering non-existent the integral in the original equation (22). To prevent this we must choose A' so as to cancel the pole at $\zeta = \alpha_3$. This requires

$$A' = -\alpha_3, \quad \text{i.e.} \quad A = -\alpha_3\,\mathrm{e}^{k_0}/\mathrm{i}\nu\pi^{\frac{1}{2}}, \tag{65}$$

and we are left with

$$\bar{h}(\zeta) = \mathrm{e}^{-Z(\zeta)}/(\zeta-\alpha_2). \tag{66}$$

If $\nu > 2$ however, the extra factors $(\zeta-\beta_1)(\zeta-\beta_2)$ in the denominator give rise to two further poles of $\bar{h}(\zeta)$. That at β_1 being in the right-hand half plane cannot be allowed, but since no further freedom exists to cancel it, we must conclude that for $\nu > 2$ no solution of (22) is possible for any value of A.†

With $\bar{h}(\zeta)$ given by (66), the contour of integration in (64) can be deformed to the two sides of the negative real axis, the pole at $\zeta = \alpha_2$ being passed over in the process. The integral along the axis is evaluated with the aid of the Plemelj formula for $Z(\zeta)$, namely

$$\left.\begin{aligned} \xi < 0: \quad Z^{\pm}(\xi) &= \Theta(\xi) \pm \mathrm{i}[\theta(\xi)-\pi], \\ \text{where} \qquad \Theta(\xi) &= \frac{1}{\pi}\int_{-\infty}^{0}\frac{\theta(\xi_1)-\pi}{\xi_1-\xi}\,\mathrm{d}\xi_1. \end{aligned}\right\} \tag{67}$$

† But see footnote on p. 472.

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Using this result we find

$$\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{(0+)} \bar{h}(\zeta)\, \mathrm{e}^{\zeta x} \mathrm{d}\zeta/\zeta = -\frac{1}{\pi}\int_{-\infty}^{0} \frac{\sin\theta(\xi)\, \mathrm{e}^{-\Theta(\xi)+\xi x}\, \mathrm{d}\xi}{\xi(\xi-\alpha_2)} = i(x) \tag{68}$$

say. The residue at α_2 is

$$\alpha_2^{-1}\exp\left[-Z(\alpha_2)+\alpha_2 x\right]. \tag{69}$$

This can also be written $-i(0)\,\mathrm{e}^{\alpha_2 x}$, as is seen by integrating $\zeta^{-1}\bar{h}(\zeta)$ round a contour consisting of the infinite circle together with the two sides of the negative real axis. Adding the contributions, therefore,

$$h(x) = i(x) - i(0)\,\mathrm{e}^{\alpha_2 x}. \tag{70}$$

Since the real part of α_2 is negative, the last term decays exponentially as $x \to \infty$.

4. Discussion of deflexion solution

4·1. *Limiting behaviour of $h(x)$ for small and large values of x*

(i) The behaviour for small x follows from that of $\bar{h}(\xi)$ for large ξ. To find this we note from (39) and (44) that $\Omega(\xi) \sim -(\ln\xi)/\pi\xi$, whence, by (50),

$$\bar{h}(\xi) \sim \frac{1}{\xi} - \frac{\ln\xi}{\pi\xi^2} + O(\xi^{-2}), \tag{71}$$

and termwise inversion gives

$$h(x) = x + \tfrac{1}{2}(x^2/\pi)\ln x + O(x^2). \tag{72}$$

which clearly satisfies the boundary conditions (21) and is consistent with (28). The two terms displayed are independent of ν, being precisely those of the steady solution.

(ii) Apart from the term which decays exponentially, $h(x)$ behaves like $i(x)$ when x is large. $i(x)$ is given by (68) as a Laplace integral, and its asymptotic behaviour as $x\to\infty$ can be inferred from that of the integrand near $\xi = 0$. For small $|\xi|$, $\Theta(\xi)$ can be expanded, after integration by parts, in a form similar to (48), namely

$$\left.\begin{aligned} &\Theta(\xi) = -\tfrac{1}{2}\ln|\xi| + l_0(\nu) + \xi l_1(\nu) + \tfrac{1}{2}\xi^2 l_2(\nu) + O_l(\xi^3),\\ \text{where}\quad &l_0 = -\frac{1}{\pi}\int_{-\infty}^{0}\theta'(\xi_1)\ln|\xi_1|\,\mathrm{d}\xi_1,\quad l_n = \frac{1}{\pi}\int_{-\infty}^{0}\theta'(\xi_1)\,\mathrm{d}\xi_1/\xi_1^n \quad (n=1,2)\end{aligned}\right\} \tag{73}$$

and $O_l(\xi^3)$ denotes terms that are of order ξ^3 except possibly for multiplication by powers of $\ln\xi$. Insertion of this expansion and that of $(\xi-\alpha_2)^{-1}$ in (68) shows that

$$\left.\begin{aligned} &i(x) \sim -\frac{\mathrm{e}^{-l_0}}{\pi\alpha_2}\int_{-\infty}^{0}(1+\lambda_1\xi+\lambda_2\xi^2+O_l(\xi^3))\,\mathrm{e}^{\xi x}\,\mathrm{d}\xi/|\xi|^{\frac{1}{2}},\\ \text{where}\quad &\lambda_1 = \frac{1}{\alpha_2} - l_1,\quad \lambda_2 = \tfrac{1}{2}(l_1^2 - l_2) - \frac{l_1}{\alpha_2} + \frac{1}{\alpha_2^2}.\end{aligned}\right\} \tag{74}$$

The factor outside the integral can be replaced by $A/\mathrm{i}\nu\pi^{\frac{1}{2}}$, as follows. As $\zeta\to 0$ from a point in the lower-half plane, (56) becomes

$$-(k_0+l_0) + \tfrac{1}{2}\mathrm{i}\pi = -\ln(-\alpha_1), \tag{75}$$

and when use is made of the fact that $\alpha_1\alpha_2\alpha_3 = -\mathrm{i}\nu^2$, and the value of A is inserted from (65), we obtain

$$-\frac{\mathrm{e}^{-l_0}}{\pi\alpha_2} = \frac{\alpha_3\,\mathrm{e}^{k_0}}{\pi\nu^2} = \frac{A}{\mathrm{i}\nu\sqrt{\pi}}.$$

With this substitution, and carrying out the integration in (74), we finally obtain the asymptotic expansion

$$h(x) \sim i(x) \sim \frac{A}{\mathrm{i}\nu x^{\frac{1}{2}}}\left[1-\frac{\lambda_1}{2x}+\frac{3\lambda_2}{4x^2}+O_l\left(\frac{1}{x^3}\right)\right], \tag{76}$$

the difference between $h(x)$ and $i(x)$ being the exponentially small term (69). The limiting result that $x^{\frac{1}{2}}h(x) \to A/\mathrm{i}\nu$ as $x \to \infty$ agrees with (23).

Coefficients in asymptotic expansion

The quantities l_0, l_1 and l_2 defined by (73) are found in terms of the roots α_i, β_i by direct integration using the expression (55) for $\theta'(\zeta)$. The result is

$$\left.\begin{aligned} l_0 &= \frac{1}{4\pi\mathrm{i}}\sum_{i=1}^{3}[(\ln\alpha_i)^2-(\ln\beta_i)^2];\\ n=1,2:\quad l_n &= -\frac{1}{2\pi\mathrm{i}}\sum_{i=1}^{3}\left[\frac{\ln\alpha_i}{\alpha_i^n}-\frac{\ln\beta_i}{\beta_i^n}\right].\end{aligned}\right\} \tag{77}$$

In obtaining the second result, use has been made of the fact, noted in appendix B, that

$$\Sigma\left(\frac{1}{\alpha_i}-\frac{1}{\beta_i}\right) = \Sigma\left(\frac{1}{\alpha_i^2}-\frac{1}{\beta_i^2}\right) = 0.$$

Consideration of the location of the roots, as indicated in appendix C, enables us to simplify one part of each of these as follows

$$\left.\begin{aligned} \mathscr{R}l_0 &= \tfrac{1}{2}\ln|\alpha_1|,\\ \mathscr{I}l_1 &= \tfrac{1}{2}|\alpha_1|^{-1} = \mathscr{I}\frac{1}{\alpha_2}-\frac{1}{\nu},\\ \mathscr{R}l_2 &= \tfrac{1}{2}|\alpha_1|^{-2}.\end{aligned}\right\} \tag{78}$$

When the first of these latter is substituted in (75) we obtain

$$A(\nu) = (1/\surd\pi)\,\mathrm{e}^{\mathrm{i}\phi(\nu)},\quad \text{say}, \tag{79}$$

where $\phi(\nu) = -(\arg\alpha_2+\mathscr{I}l_0+\frac{1}{2}\pi)$ is real. Thus $|A(\nu)| = 1/\surd\pi$ for all ν.

It may also be noted for later use that a relation similar to (75) exists between l_1, l_2 and the corresponding coefficients k_1, k_2 in the expansion (48), namely

$$k_n+l_n = -1/a_1^n \quad (n = 1, 2). \tag{80}$$

4·2. *Lift force*

On substitution from (18) into (8), with $w_W = 0$, the expression found for the lift force is

$$\left.\begin{aligned} L &= 2\rho U^2 c\tau\pi\mu^{\frac{1}{2}}[(1+\tfrac{1}{2}\mathrm{i}\omega)A(\nu)-\mathrm{i}\nu B(\nu)]\exp(\mathrm{i}\omega Ut/c),\\ \text{where}\quad B(\nu) &= \frac{1}{\pi}\int_0^\infty x^{\frac{1}{2}}g(x)\,\mathrm{d}x,\end{aligned}\right\} \tag{81}$$

and $A(\nu)$ is given by (23). $B(\nu)$ can be written in terms of quantities already defined, by rearranging (24) as

$$\left(\mathrm{i}\nu+\frac{\mathrm{d}}{\mathrm{d}x}\right)h(x) = -\frac{1}{\pi}\int_0^\infty\left(\frac{x_1}{x}\right)^{\frac{1}{2}}\frac{g(x_1)\,\mathrm{d}x_1}{x_1-x}+\frac{A}{x^{\frac{1}{2}}}.$$

As $x \to \infty$, the right-hand side behaves asymptotically like $B/x^{\frac{3}{2}} + A/x^{\frac{1}{2}}$. Comparing this with the asymptotic behaviour of the left-hand side, obtained from (76), we find that

$$\nu B(\nu)/A(\nu) = \tfrac{1}{2}\mathrm{i}(1+\mathrm{i}\nu\lambda_1) = \Lambda_1(\nu), \quad \text{say} \tag{82}$$

$\Lambda_1(\nu)$ is pure real since, by the second result of (78), $\mathscr{I}\lambda_1 = 1/\nu$. With the insertion of (79), the expression for lift is thus

$$L = L_0[1+\tfrac{1}{2}\mathrm{i}\omega - \mathrm{i}\Lambda_1(\nu)] \exp \mathrm{i}\left(\frac{\omega U t}{c} + \phi(\nu)\right), \tag{83}$$

where
$$L_0 = (\tfrac{1}{2}\rho U^2 c)\, 4\tau(\pi\mu)^{\frac{1}{2}} \tag{84}$$

is the value obtained by Spence (1961 *a*) for the lift force in steady flow with deflexion-angle τ, in the limiting case of small μ.

$\phi(\nu)$ and $\Lambda_1(\nu)$ are plotted in figure 6 and tabulated on page 469. In the limit of small ν, both quantities behave like $\nu^{\frac{1}{2}}$, so the limiting situation as $\omega \to 0$ is one in which the lift force simply oscillates with reduced frequency ω and amplitude equal to its value L_0 in steady flow.

The lift force has been calculated from (83) as a function of ω for two particular values of μ, namely 0·025 and 0·25. The results are shown in figure 8 as plots of $|L/L_0|$ and of the phase advance. It is seen that $|L/L_0|$ does not begin to depart appreciably from unity until ω is greater than 1.

4·3. *Behaviour when ν is small*

To discuss the behaviour of the solution as $\nu \to 0$ we calculate the limiting form of $\Theta(\xi)$ explicitly, using the expansions of α_i and β_i already quoted in equation (62). The lines of the calculation are indicated in appendix D. The result can be written in the form

$$\left.\begin{aligned} \Theta(\xi) &= \Theta_0(\xi) + \frac{\mathrm{i}\nu}{\pi}\left\{\frac{2\ln|\xi| - \pi\xi}{\xi^2+1} - \Psi\left(\frac{\xi}{\nu}\right)\right\} + O(\nu^2 \ln \nu), \\ \text{where} \quad \Theta_0(\xi) &= -\frac{1}{\pi}\int_{-\infty}^{0} \frac{\cot^{-1}|\xi|\, \mathrm{d}\xi_1}{\xi_1 - \xi}, \end{aligned}\right\} \tag{85}$$

which is the value of $\Theta(\xi)$ when $\nu = 0$. The function Ψ is given by equation (D 8). The integral (68) is divergent in the limit $\nu = 0$, since the integrand then behaves like $\xi^{-\frac{3}{2}}(\xi+\alpha_2) \sim \xi^{-\frac{1}{2}}$, but we can still calculate the reduced downwash, which is given by

$$h'(x) + \mathrm{i}\nu h(x) = -\frac{1}{\pi}\int_{-\infty}^{0} \sin\theta(\xi) \exp\left[-\Theta(\xi) + \xi x\right] \frac{\mathrm{d}\xi}{\xi}. \tag{86}$$

On insertion of the above expansion for $\Theta(\xi)$, together with that of $\sin\theta(\xi)$, namely

$$\sin\theta(\xi) = (\xi^2+1)^{-\frac{1}{2}}\left[1 + \frac{2\mathrm{i}\nu\xi}{\xi^2+1} + O(\nu^2)\right], \tag{87}$$

the right-hand side of (86) can be written uniformly in x in the form

$$f_0(x)\,[1+O(\nu)], \tag{88}$$

say, where
$$f_0(x) = -\frac{1}{\pi}\int_{-\infty}^{0} \frac{\exp\left[-\Theta_0(\xi) + \xi x\right]}{\xi(\xi^2+1)^{\frac{1}{2}}}\,\mathrm{d}\xi. \tag{89}$$

This function is precisely that found by Spence (1961 *a*) as the downwash distribution in steady flow. It satisfies the equation

$$f_0(x) = \frac{1}{\pi}\int_0^{\infty}\left(\frac{x}{x_1}\right)^{\frac{1}{2}} \frac{f_0'(x_1)\,\mathrm{d}x_1}{x_1 - x}, \quad f_0(0) = 1, \tag{90}$$

and behaves like $(\pi x)^{-\frac{1}{2}}$ as $x \to \infty$. (The equation (90) was first solved by Lighthill (1959) who wrote $f_0(x)$ as a Mellin integral, and by Stewartson (1959) who used the Wiener–Hopf technique to obtain $f_0(x)$ as a Fourier transform.)

Combining (86) and (88) and solving with the terms of order ν excluded, we obtain

$$h(x) = \int_0^x e^{i\nu(x_1-x)} f_0(x_1)\, dx_1. \tag{91}$$

If $\nu x \ll 1$ the exponential can be treated as unity, and we have simply

$$h(x) = \int_0^x f_0(x_1)\, dx_1, \tag{92}$$

which can be seen by inspection to satisfy equations (24) and (25) when ν is set to zero at the outset. This is the solution suggested in Spence (1961 *b*) for the low-frequency case, and we now see that it is an approximation to the full solution only up to non-dimensional distances from the trailing edge $x = 0$ such that

$$x \ll 1/\nu, \quad \text{i.e.} \quad \mu c x \ll c/\omega, \tag{93}$$

that is at physical distances that are small compared with the $(2\pi)^{-1}$ times the wavelength of the oscillation (namely the distance $2\pi(c/\omega)$ travelled by a particle in the undisturbed stream during a complete cycle). Within this range the first term on the left of (86) dominates over the second and the approximation $\nu = 0$ is permissible. However the corresponding value of $h(x)$ behaves like $2(x/\pi)^{\frac{1}{2}}$ as $x \to \infty$, which is physically unrealistic since it would imply a finite value of h_J at infinity. But if the second term on the left of (86) is retained it ultimately becomes dominant, no matter how small the value of ν, at distances $x \gg 1/\nu$. The limiting form of solution is then found as an asymptotic expansion by partial integration of (91), in the form

$$h(x) \sim (i\nu)^{-1} f_0(x) \sim \frac{1}{i\nu(\pi x)^{\frac{1}{2}}} \tag{94}$$

in agreement with (76), since $A \to \pi^{-\frac{1}{2}}$ as $\nu \to 0$.

4·4. *Improved approximation for μ less small*

The solution of the previous sections, obtained by setting $\mu = 0$ in (19), approximates to the full solution uniformly over the whole range $0 < x < \infty$. It can be improved by replacing $\mu/(1+\mu x)$ in (19) by μ, and retaining this term but excluding its square. This procedure is not uniformly valid since when $x \gg 1/\mu$, $\mu/(1+\mu x)$ behaves instead like $1/x$ which is then $\ll \mu$; in effect it is an 'inner' expansion of the full solution, and a different form of approximation would be needed to treat the 'outer' region $x \gg 1/\mu$. It is, however, adequate for calculating the next approximation (in μ) to the lift force.

To the stated approximation, (19) is

$$\left\{\left(i\nu + \frac{d}{dx}\right)^2 - \mu\left(i\nu + \frac{d}{dx}\right)\right\} h(x) - \frac{1}{\pi}\int_0^\infty \left(\frac{x_1}{x}\right)^{\frac{1}{2}} (h''' - \tfrac{3}{2}\mu h'')\frac{dx_1}{x_1 - x} = (i\nu - \tfrac{1}{2}\mu)\frac{A}{x^{\frac{1}{2}}}. \tag{95}$$

Operations similar to those of § 3·1 show that the Laplace transform $\bar{h}(\xi)$ defined by (26) satisfies

$$\xi^{-2}(\xi + i\nu)(\xi + i\nu - \mu)\,\bar{h}(\xi) + \frac{1}{\pi}\int_0^\infty \left(\frac{\xi_1}{\xi}\right)^{\frac{1}{2}} (\xi_1 - \tfrac{3}{2}\mu)\left(\bar{h}(\xi_1) - \frac{1}{\xi_1}\right)\frac{d\xi_1}{\xi_1 - \xi} = (i\nu - \tfrac{1}{2}\mu)\, A\pi^{\frac{1}{2}} i\xi^{-\frac{3}{2}}. \tag{96}$$

The solution of this equation follows exactly the same lines as that of (33). Expressed in terms of a function

$$\theta(\xi) = \frac{1}{2\mathrm{i}}\ln\frac{\xi^2(\xi-\frac{3}{2}\mu)+\mathrm{i}(\xi+\mathrm{i}\nu)(\xi+\mathrm{i}\nu-\mu)}{\xi^2(\xi-\frac{3}{2}\mu)-\mathrm{i}(\xi+\mathrm{i}\nu)(\xi+\mathrm{i}\nu-\mu)} = \frac{1}{2\mathrm{i}}\ln\prod_{j=1}^{3}\left(\frac{\xi-\beta_j}{\xi-\alpha_j}\right), \tag{97}$$

say, which is a generalization of (36), it is

$$\bar{h}(\xi) = \frac{\xi(\xi+\mathrm{i}\nu A\pi^{\frac{1}{2}}\,\mathrm{e}^{-k_0})\,\mathrm{e}^{\Omega(\xi)}}{\sqrt{\{\xi^4(\xi-\frac{3}{2}\mu)^2+(\xi+\mathrm{i}\nu)^2(\xi+\mathrm{i}\nu-\mu)^2\}}}. \tag{98}$$

$\Omega(\xi)$ and k_0 being defined in terms of the new θ by (40) and (48). An analytic continuation identical with that of § 3·3 then enables us to write

$$\bar{h}(\zeta) = \mathrm{e}^{-Z(\zeta)}/(\zeta-\alpha_2), \tag{99}$$

as before, $Z(\zeta)$ being defined by (51) in terms of the new θ, and α_2 being the appropriate root of the new cubic. When μ and ν are small and of the same order, the roots are

$$\left.\begin{aligned}\left.\begin{matrix}\alpha_1\\\beta_1\end{matrix}\right\} &= \pm\mathrm{i}+2\mathrm{i}\nu+\tfrac{1}{2}\mu, \quad \left.\begin{matrix}\alpha_2\\\beta_2\end{matrix}\right\} = -\mathrm{i}\nu[1\pm\nu(\tfrac{3}{2}+\mathrm{i}\omega)],\\ \left.\begin{matrix}\alpha_3\\\beta_3\end{matrix}\right\} &= (\mu-\mathrm{i}\nu)\,[1\pm(\mu-\mathrm{i}\nu)\,(\tfrac{1}{2}-\omega)].\end{aligned}\right\} \tag{100}$$

Inversion of (99) shows that the asymptotic form of the solution is the same as (74) up to the term in $x^{-\frac{3}{2}}$, with coefficients that now depend on μ as well as on ν. The next term however behaves like $(\ln x)/x^{\frac{5}{2}}$ in this case, a consequence of the non-uniformity of the approximation at large x. It is straightforward but tedious to repeat the calculations of § 4·2 and appendix D to find $A(\nu,\mu)$ and $B(\nu,\mu)$ and hence the lift force. The details will not be given, but it is of interest to note that in the limit $\omega = 0$ the expression for lift takes the form

$$L_0\{1-(\mu/2\pi)\ln\mu\},$$

which agrees with that found for steady flow (Spence (1961 *a*)) by a somewhat different expansion of the downwash function $h'(x)$ in powers of μ.

A more useful exercise is to find the relation between μ and ν at the critical point beyond which a solution becomes impossible because the imaginary part of the root β_1 becomes positive. To do this we find the condition for β_1 to be real.

β_j $(j = 1, 2, 3)$ are the roots of

$$[\xi^2(\xi-\tfrac{3}{2}\mu)-2\nu\xi+\mu\nu]+\mathrm{i}\{\xi^2-\mu\xi-\nu^2\} = 0.$$

If β is a *real* root of this equation, the expressions in the square and in the curly brackets both vanish when $\xi = \beta$. The condition for the second of these is

$$\beta = \tfrac{1}{2}\mu\pm\tfrac{1}{2}\sqrt{(\mu^2+4\nu^2)}.$$

We identify the upper and lower signs with β_1 (positive) and β_2 (negative) respectively. On substitution of $\xi = \beta_1$ in the square bracket, the condition that the latter should vanish is found after a little algebra to be

$$(\nu^2-2\nu-\tfrac{1}{2}\mu^2)\sqrt{(\mu^2+4\nu^2)} = \tfrac{1}{2}\mu^3,$$

or, on setting $\nu = \mu\omega$,
$$\tfrac{1}{2}\nu = [1-(1/2\omega^2)\{1+(1+4\omega^2)^{-\frac{1}{2}}\}]^{-1}. \tag{101}$$

If this value is denoted by $\nu_{\text{crit.}}$, then for $\nu < \nu_{\text{crit.}}$ the imaginary part of β_1, the root with positive real part, is negative, and $\bar{h}(\zeta)$ has no pole at β_1, so solutions of (95) exist. If, however, $\nu > \nu_{\text{crit.}}$, $\mathscr{I}\beta_1 > 0$, and no solution is possible. This is consistent with the earlier result that $\nu_{\text{crit.}} = 2$ in the limiting case $\mu = 0$, for on setting $\omega = \nu/\mu = \infty$ in (101) we obtain the same value.

5. Computations of jet deflexion

The integral $i(x)$ has been evaluated for a number of values of ν on a Mercury computer using complex arithmetic by Miss L. Klanfer of Aerodynamics Department, R.A.E. A preliminary step was to find the root α_2 of the cubic $\xi^3 = \mathrm{i}(\xi + \mathrm{i}\nu)^2$, namely that with both real and imaginary parts negative. A standard program for solving cubic equations gave the values:

ν	0·05	0·1	0·5	1·0	1·5
$\mathscr{R}\alpha_2$	−0·00994	−0·02554	−0·17913	−0·36899	−0·54721
$\mathscr{I}\alpha_2$	−0·04677	−0·08856	−0·34131	−0·57395	−0·76709

(The remaining roots are $\alpha_3 = \mathrm{i}(\mathscr{I}\alpha_2) - \mathscr{R}\alpha_2$, $\alpha_1 = \mathrm{i}(1 - 2\mathscr{I}\alpha_2)$. To put the infinite integral (68) into a form suitable for computation, the transformation

$$\xi = -\tan^2(\tfrac{1}{2}\phi), \quad \theta(\xi) = \pi - G(\phi) \tag{102}$$

was made. The inner integral is then

$$\Theta(\xi) = -(2/\pi)(\cos^2\tfrac{1}{2}\phi)\int_0^\pi \frac{G(\phi_1)\tan\frac{1}{2}\phi_1\,\mathrm{d}\phi_1}{\cos\phi_1 - \cos\phi}. \tag{103}$$

This was evaluated by interpolating to $G(\phi)$ by a series

$$G(\phi) = \left(\tan^2\tfrac{1}{2}\phi + \frac{2}{\pi}\right)^{-1} + \cot\tfrac{1}{2}\phi\sum_{n=0}^{N} a_n\cos n\phi, \tag{104}$$

the first term on the right being chosen to have the correct behaviour at both 0 and π in order to improve the convergence of the Fourier coefficients a_n. By trial and error it was found satisfactory for ν up to 1 to work with $N = 45$, when the last 10 or so coefficients were 10^{-4} times the largest. A larger number would, however, be necessary to achieve the same accuracy for $\nu = 1{\cdot}5$. The expression (103) is then equal to

$$-\ln(\tan\tfrac{1}{2}\phi) - \cot\tfrac{1}{2}\phi\sum_{n=1}^{N}\sin n\phi + J(\phi), \tag{105}$$

where $$J(\phi) = [\tfrac{1}{2}\tan^2\tfrac{1}{2}\phi\ln(\tan\tfrac{1}{2}\phi) - (1/\pi)\ln\tfrac{1}{2}\pi]\,[\tan^2\tfrac{1}{2}\phi + (2/\pi)]^{-1}$$
and altogether

$$i(x) = -\frac{1}{\pi}\int_0^\pi \exp\left\{\cot\tfrac{1}{2}\phi\sum_{n=1}^{N} a_n\sin n\phi - J(\phi) - x\tan^2\tfrac{1}{2}\phi\right\}\frac{\sin G(\phi)\,\mathrm{d}\phi}{\sin^2\frac{1}{2}\phi + \alpha_2\cos^2\frac{1}{2}\phi}, \tag{106}$$

a_n, G, J and α_2 in this expression are all complex. The integral was evaluated by Gaussian integration of the real and imaginary parts separately. Finally $h(x)$ was found from $i(x)$ with the aid of equation (70). A check on the accuracy of computation was provided by a separate evaluation of $i(0)$ as $\alpha_1^{-2}\exp\{Z(\alpha_2)\}$.

A further check is provided by the value of $|A\surd\pi|$, which is known by (79) to be unity, and which can be expressed in terms of the Fourier coefficients as

$$A\surd\pi = -\frac{i\nu}{\alpha_2}\exp\left[-\lim_{\xi\to 0}\{\Theta(\xi)+\tfrac{1}{2}\ln|\xi|\}\right] = -\frac{i\nu}{\alpha_2}\surd(\tfrac{1}{2}\pi)\exp\left(2\sum_{n=1}^{N} na_n\right). \tag{107}$$

Values of $|A\surd\pi|$ computed from this expression are

ν	0	0·05	0·1	0·5	1·00	1·50
$\lvert A\surd\pi\rvert$	1	0·9997	0·9997	0·9997	1·0009	1·0055

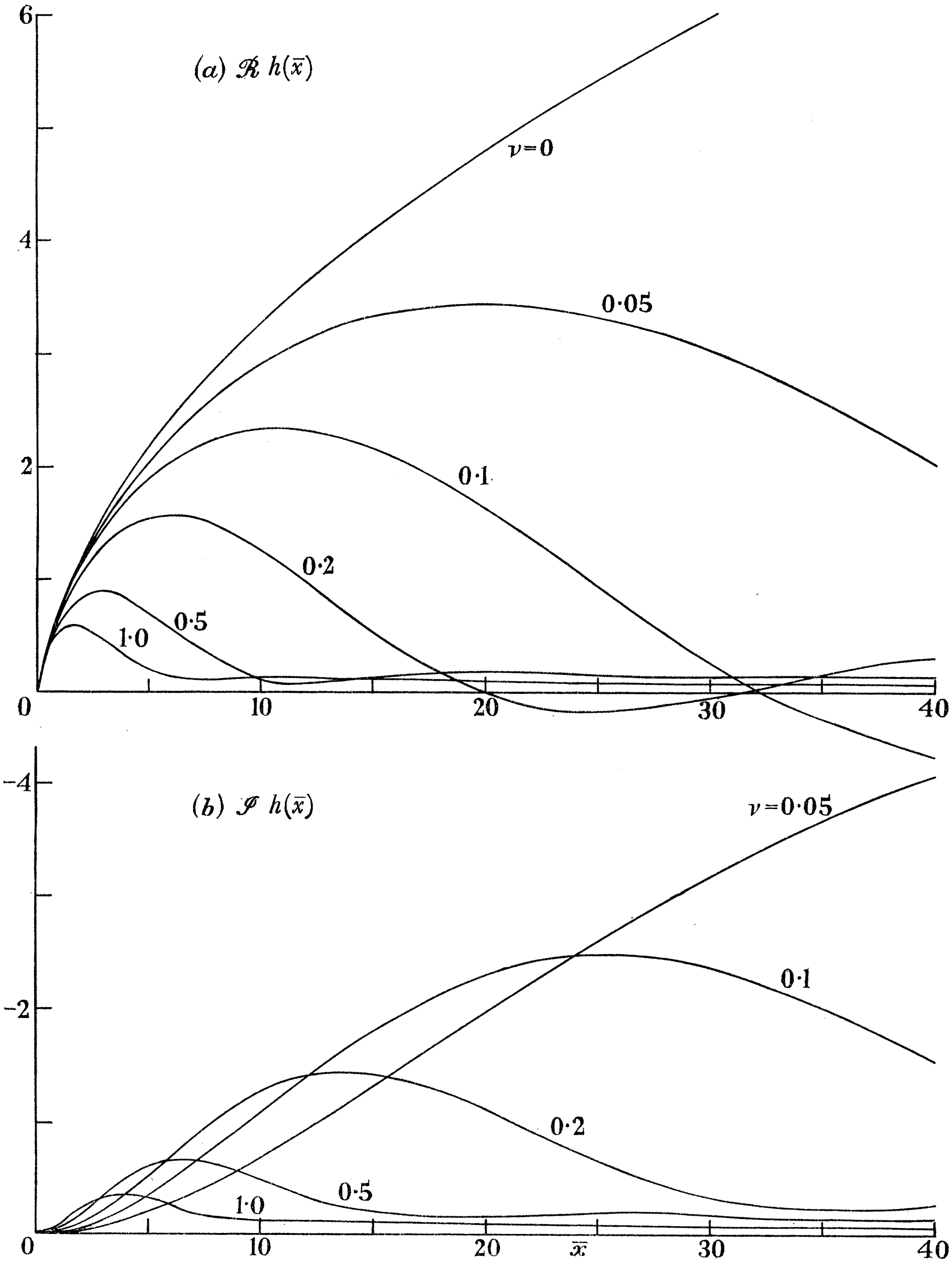

FIGURE 4. Computed values of $h(\bar{x})$: (a) Real part; (b) Imaginary part.

showing that the accuracy of calculation with $N = 45$ is good up to $\nu = 1$ but begins to deteriorate beyond this point.

The results, in the form of plots of the real and imaginary parts of $h(x)$ for these values of ν and x up to 10 are shown in figure 4. It will be noted that the amplitudes are still increasing at $x = 10$ for $\nu = 0{\cdot}05$ and $0{\cdot}1$. The curves do, however, ultimately decrease, and

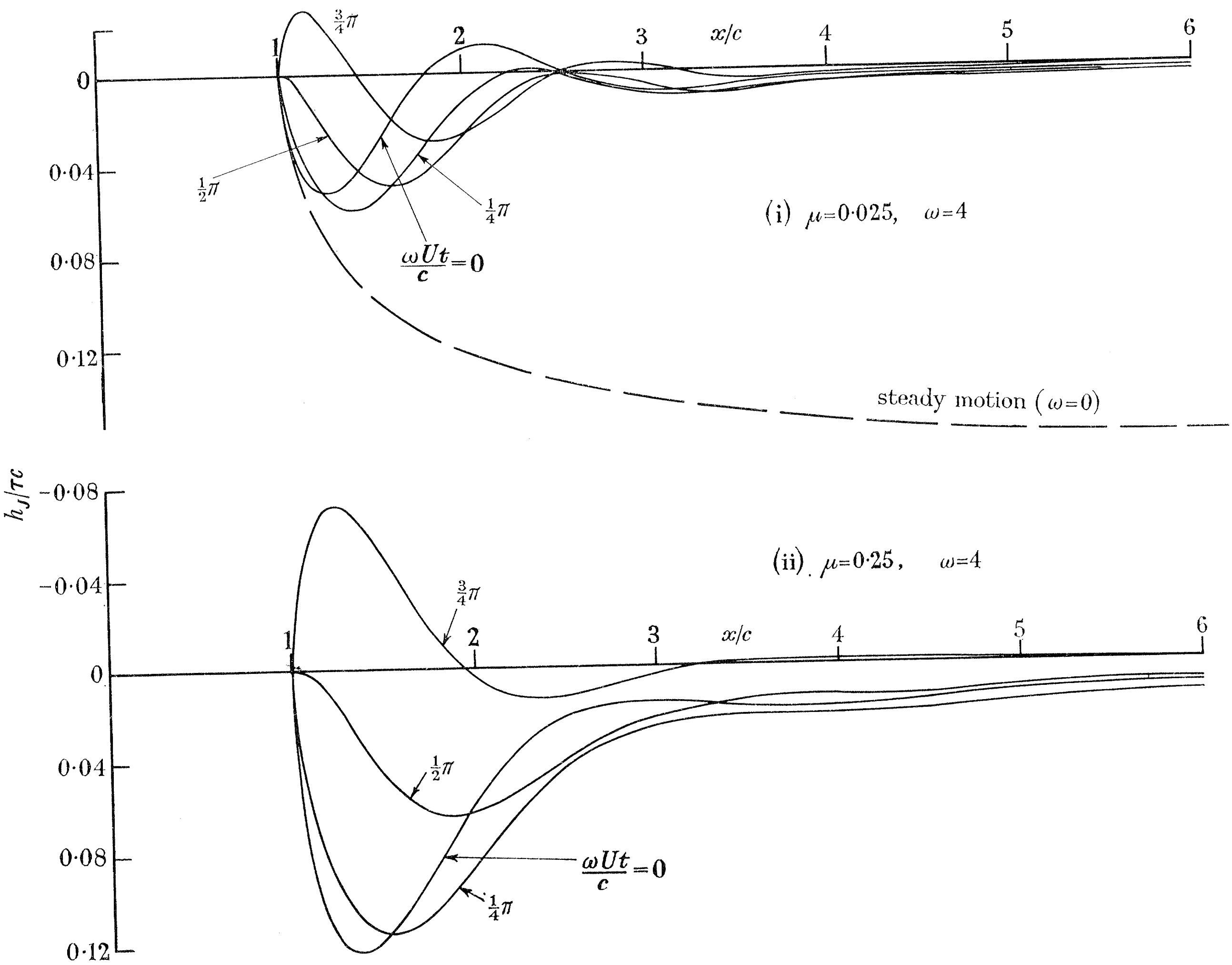

FIGURE 5. Instantaneous jet shapes in deflexion case for two values of ν.

the asymptotic expression (76) was found to be very accurate in all cases for x between 50 and 100. For larger values of x it is more accurate than numerical integration of (106). The curve for $\nu = 0$ shown in the upper part of figure 4 is obtained by integration of the expression

$$h(x) = \int_0^x f_0(x)\,\mathrm{d}x,$$

where f_0 is defined by (89), which applies in this case. Since $f_0(x) \sim (\pi x)^{-\frac{1}{2}}$, $h(x)$ in this case continues to grow indefinitely like $2(x/\pi)^{\frac{1}{2}}$, and as remarked in § 4·3, the other curves also behave in this way when x is small compared with ν^{-1}.

The instantaneous location $h_J(x)$ of the jet, which is given by the real part of $h_J(x)\,\mathrm{e}^{\mathrm{i}\omega Ut/c}$ (x now being the physical distance $c(1+\mu\bar{x})$), for $\omega = 4$ and for two particular values of μ, namely 0·025 and 0·25 (i.e. $C_J = 0{\cdot}1$ and 1·0) is plotted in figure 5. The curves are derived from the non-dimensional curves of $h(\bar{x})$ for $\nu = 0{\cdot}1$ and $\nu = 1{\cdot}0$, using equation (18), and are shown for the phase angles $\omega Ut/c = 0, \frac{1}{4}\pi, \frac{1}{2}\pi$ and $\frac{3}{4}\pi$.

6. Plunging motion of the wing, with jet tangential

In this and the next section respectively we derive the solutions for cases in which the wing: (i) moves up and down sinusoidally with amplitude βc say, β being small compared with 1, while remaining parallel to the undisturbed stream, as indicated in figure 1 (*b*)—this corresponds to motion through a 'row' of gusts (or through a single frequency component of the turbulent spectrum); and (ii) oscillates in incidence with amplitude α about the trailing edge. In either case we shall treat the jet-deflexion at the trailing edge as zero, that is set

$$\left(\frac{\partial h_J}{\partial x}\right)_{x=c} = \left(\frac{\partial h_W}{\partial x}\right)_{x=c}, \tag{108}$$

since the solution already obtained need merely be linearly superposed to deal with motions in which the deflexions are not zero.

6·1. *Governing equations*

For the plunging motion treated in this section we set

$$h_W = \beta c\, \mathrm{e}^{\mathrm{i}\omega Ut/c}, \quad \text{whence} \quad w_W = \mathrm{i}\omega U\beta\, \mathrm{e}^{\mathrm{i}\omega Ut/c}. \tag{109}$$

On substitution of this expression for w_W the right-hand side of (3), which was zero in the deflexion case, becomes

$$\mathrm{i}\omega U\beta\left[1-\left(\frac{x-c}{x}\right)^{\frac{1}{2}}\right]\mathrm{e}^{\mathrm{i}\omega Ut/c}. \tag{110}$$

Equations (4) and (6) remain as before. If now we set

$$\left.\begin{aligned} h_J &= \beta c[1+\mathrm{i}\nu(\mu c/x)^{\frac{1}{2}}h_1(\bar{x})]\,\mathrm{e}^{\mathrm{i}\omega Ut/c},\\ \gamma_J &= 2\mathrm{i}\omega U\beta(\mu c/x)^{\frac{1}{2}}g_1(\bar{x})\,\mathrm{e}^{\mathrm{i}\omega Ut/c},\end{aligned}\right\} \tag{111}$$

where $\bar{x}$ is defined by (17) as before, then by combining (3), (4) and (6) we obtain the equations

$$\left(\mathrm{i}\nu+\frac{\mathrm{d}}{\mathrm{d}\bar{x}}\right)h_1(\bar{x}) = -\frac{1}{\pi}\int_0^\infty\left(\frac{\bar{x}}{\bar{x}_1}\right)^{\frac{1}{2}}\frac{g_1(\bar{x}_1)\,\mathrm{d}x_1}{\bar{x}_1-\bar{x}}-\bar{x}^{\frac{1}{2}}, \tag{112}$$

$$\left(\mathrm{i}\nu+\frac{\mathrm{d}}{\mathrm{d}\bar{x}}\right)g_1(\bar{x}) = -h_1'''(\bar{x}), \tag{113}$$

which correspond to (24) and (25), the second members of each pair being identical in form. Here, as before, terms of order μ have been excluded but $\nu = \mu\omega$ retained. These may be combined as follows into a single equation for h_1. The bar over x will be omitted from now on. The right-hand side of (112) can be rewritten

$$\left.\begin{aligned} &-\frac{1}{\pi}\int_0^\infty\left(\frac{x_1}{x}\right)^{\frac{1}{2}}\frac{g_1(x_1)\,\mathrm{d}x_1}{x_1-x}+\frac{A_1}{x^{\frac{1}{2}}}-x^{\frac{1}{2}},\\ \text{where}\qquad & A_1 = \frac{1}{\pi}\int_0^\infty x_1^{-\frac{1}{2}}g_1(x_1)\,\mathrm{d}x_1,\end{aligned}\right\} \tag{114}$$

and differentiation of (112) after writing the integral in the form

$$-\frac{1}{\pi}\int_0^\infty u^{-\frac{1}{2}}g_1(xu)\,\mathrm{d}u/(u-1)$$

gives
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{i}\nu+\frac{\mathrm{d}}{\mathrm{d}x}\right)h_1(x) = -\frac{1}{\pi}\int_0^\infty\left(\frac{x_1}{x}\right)^{\frac{1}{2}}\frac{g_1'(x_1)\,\mathrm{d}x_1}{x_1-x}-\tfrac{1}{2}x^{-\frac{1}{2}}.$$

When $\mathrm{i}\nu$ times (114) is added to this, the result is

$$\mathscr{L}h_1(x) = (\mathrm{i}\nu A_1-\tfrac{1}{2})\,x^{-\frac{1}{2}}-\mathrm{i}\nu x^{\frac{1}{2}} \tag{115}$$

with boundary conditions $\quad h_1(0) = h_1'(0) = 0,$

$\mathscr{L}$ being the integro-differential operator defined by (22).

6·2. *Solution of equation* (115)

Equation (115) is of the same form as (22) except that the extra term $-\mathrm{i}\nu x^{\frac{1}{2}}$ appears on the right-hand side and the boundary condition on $h_1'(0)$ is different. The equation can be solved by the same operational technique as used in § 3. Without going through this, however, it is possible to express the solution directly in terms of that of (22) as follows. The solution of (22) will be denoted from now on by $h_0(x)$, and the corresponding eigenvalue by A_0.

Differentiation of (115) with respect to x gives

$$\mathscr{L}h_1'(x) = -\frac{\mathrm{i}\nu}{2x^{\frac{1}{2}}}+\frac{C}{x^{\frac{3}{2}}}, \tag{116}$$

where
$$C = \frac{1}{\pi}\int_0^\infty x_1^{\frac{1}{2}}h_1^{(\mathrm{iv})}(x_1)\,\mathrm{d}x_1-\tfrac{1}{2}(\mathrm{i}\nu A_1-\tfrac{1}{2}).$$

Now comparing (116) with (22) we see that the two would be identical in form if C were zero. We therefore write down the solution of (116), namely $h_1'(x) = -(1/2A_0)\,h_0(x)$, assuming C to be zero, and subsequently prove that it is so. This establishes the existence of a solution, but its uniqueness can only be shown by going through an argument like that of § 3, which will not be done here. To satisfy the boundary conditions we must choose

$$h_1(x) = -(2A_0)^{-1}\int_0^x h_0(x_1)\,\mathrm{d}x_1. \tag{117}$$

The corresponding eigenvalue A_1 is found from the asymptotic expansion of $h_1(x)$. Use of (76) shows that this is

$$h_1(x) \sim -\frac{x^{\frac{1}{2}}}{\mathrm{i}\nu}\left[1+\frac{\lambda_1}{2x}-\frac{\lambda_2}{4x^2}+\ldots\right] \tag{118}$$

as $x\to\infty$. When this is inserted in (115), equating the second terms (those in $x^{-\frac{1}{2}}$) in the asymptotic expansion of each side shows that

$$A_1 = -\tfrac{1}{2}(\lambda_1+1/\mathrm{i}\nu). \tag{119}$$

From (82) we see that $A_1 = B_0/A_0 = \nu^{-1}\Lambda_1(\nu)$. The integral in the expression for C is found from a similar calculation based on equation (22) with use of the asymptotic expansion (76); this gives

$$\frac{1}{\pi}\int_0^\infty x^{\frac{1}{2}}h_0'''(x)\,\mathrm{d}x = A_0(1+\tfrac{1}{2}\mathrm{i}\nu\lambda_1),$$

whence
$$C = -(2\pi A_0)^{-1}\int_0^\infty x^{\frac{1}{2}} h_0'''(x)\,\mathrm{d}x - \tfrac{1}{2}(\mathrm{i}\nu A_1 - \tfrac{1}{2}) = 0, \tag{120}$$

which demonstrates the consistency of the solution.

It may be noted that $h_J \to 0$ as $x \to \infty$, as would be expected on physical grounds; substituting the asymptotic expansion (118) into (111) we obtain

$$h_J(x,t) \sim \beta c\, \mathrm{e}^{\mathrm{i}\omega Ut/c}\left[1 - \left(\frac{\mu\bar{x}}{1+\mu\bar{x}}\right)^{\frac{1}{2}}\left(1 + \frac{\lambda_1}{2\bar{x}}\right)\right] = O\left(\frac{1}{x}\right), \tag{121}$$

as $\mu\bar{x} \to \infty$.

We also note that the curvature of the jet at the trailing edge, $-h_1''(0)$, is $1/2A_0$, whose magnitude is $\frac{1}{2}\sqrt{\pi}$ for all values of ν.

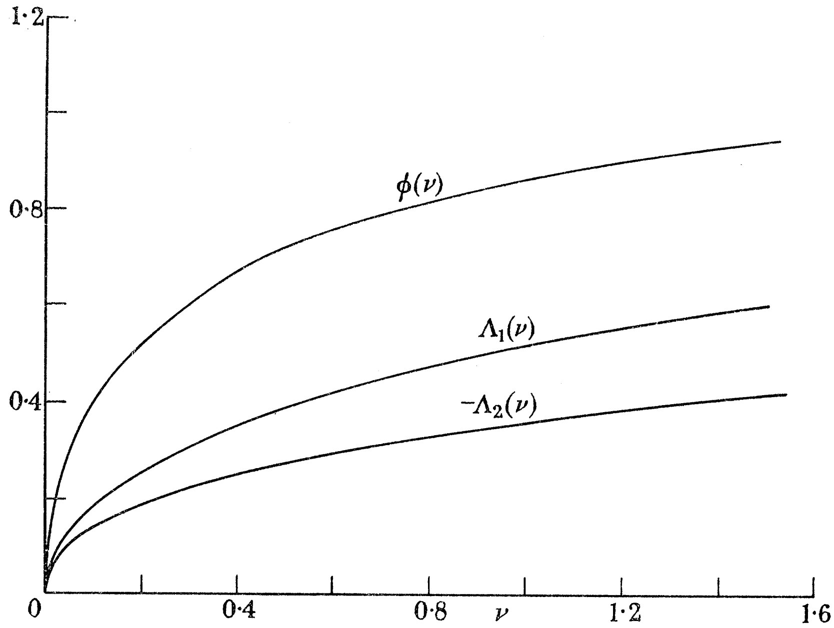

FIGURE 6. Quantities occurring in expression for lift force. ($\phi(\nu)$ is measured in radians).

Computed values

$h_1(\bar{x})$ was computed for $\nu = 0{\cdot}1$ and $1{\cdot}0$ by numerical quadrature of (117) for values of $\bar{x}$ up to 50. Above this it was necessary to integrate the asymptotic expansion of $h_0(\bar{x})$. From these results the instantaneous jet position has been computed for the same values of ω and μ as used in figure 5, namely $\omega = 4$ and $\mu = 0{\cdot}025$ and $0{\cdot}25$ and for four phase angles. The results are plotted in figure 7.

6·3. *Lift force in plunging motion*

Substitution of (109) and (111) in (9) gives

$$\Gamma_\infty(t) = \mathrm{i}\omega\beta c U\pi(1 + 2\mu A_1)\,\mathrm{e}^{\mathrm{i}\omega Ut/c},$$

and the expression (8) for the lift force becomes

$$L = \pi\rho U^2\beta c[\mathrm{i}\omega - \tfrac{3}{4}\omega^2 + (\mathrm{i} - \tfrac{1}{2}\omega)\,2\nu A_1 + 2\nu^2 B_1]\,\mathrm{e}^{\mathrm{i}\omega Ut/c}, \tag{122}$$

where
$$B_1 = \frac{1}{\pi}\int_0^\infty x^{\frac{1}{2}} g_1(x)\,\mathrm{d}x.$$

B_1 is found by substituting the asymptotic expansion (118) into (112), the right-hand side of which behaves like $B_1/x^{\frac{3}{2}} + A_1/x^{\frac{1}{2}} - x^{\frac{1}{2}}$ as $x \to \infty$. On equating the terms in $x^{-\frac{3}{2}}$ on either side we find
$$B_1 = \tfrac{1}{4}\left(\lambda_2 + \frac{\lambda_1}{\mathrm{i}\nu}\right). \tag{123}$$

We have already noted that $A_1 = \nu^{-1}\Lambda_1(\nu)$, and it is also possible to show by substitution from (74) and use of (78) that
$$\mathscr{R}\,B_1 = \tfrac{1}{2}\nu^{-2}\Lambda_1^2.$$

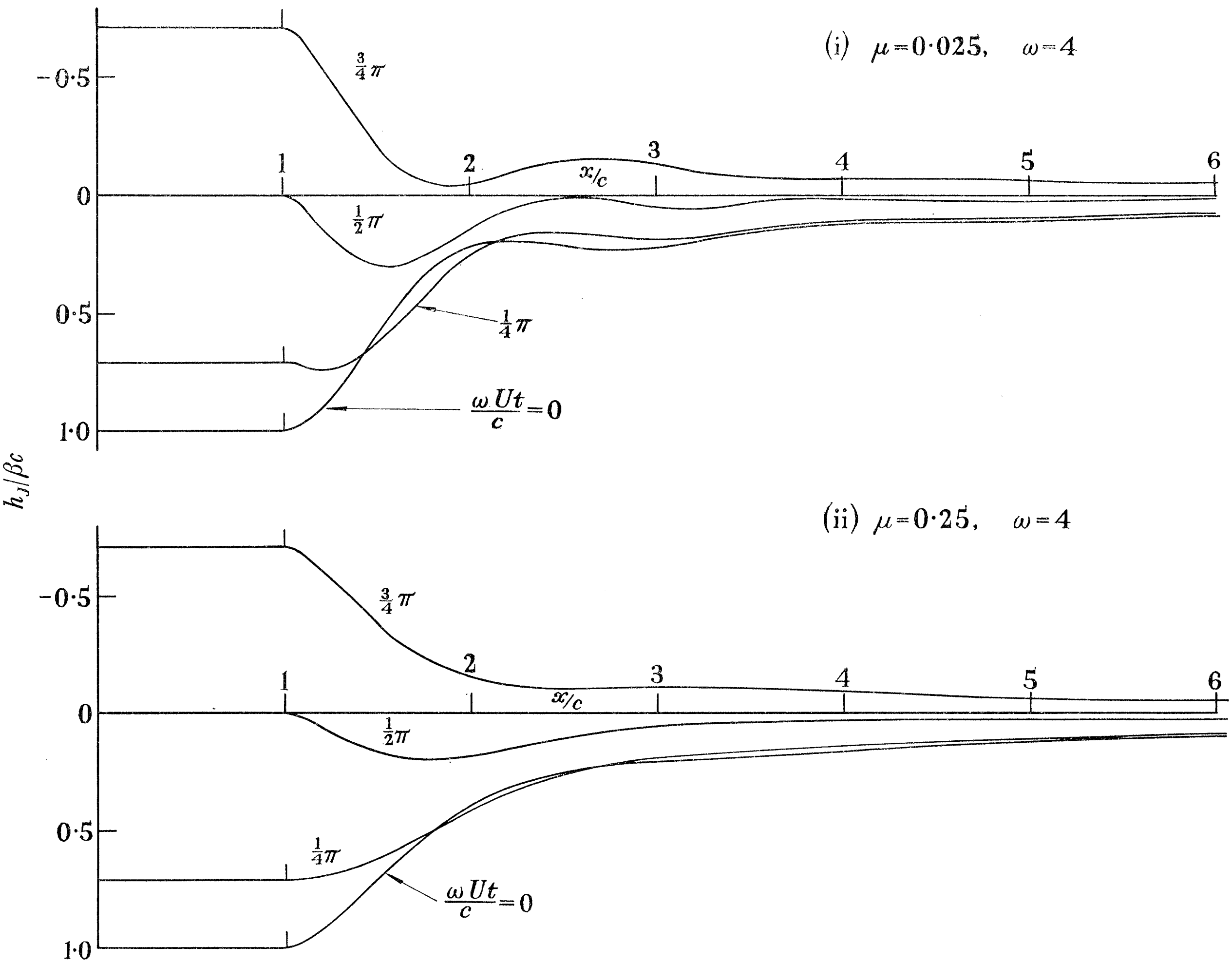

FIGURE 7. Instantaneous jet shapes in plunging motion, for two values of ν.

If the imaginary part of B_1 is written $\frac{1}{2}\nu^{-2}\Lambda_2(\nu)$, where Λ_2 is real, we can rewrite (122) as
$$L = \pi\rho U^2\beta c[\mathrm{i}\omega - \tfrac{3}{4}\omega^2 + (\Lambda_1^2 - \omega\Lambda_1) + i(2\Lambda_1 + \Lambda_2)]\,\mathrm{e}^{\mathrm{i}\omega Ut/c}. \tag{124}$$

$\Lambda_2(\nu)$ is plotted along with $\Lambda_1(\nu)$ and $\phi(\nu)$ in figure 6, and tabulated below. To prepare the table, the quantities l_0, l_1 and l_2 were first found as functions of ν from the roots α_i, β_i by means of (77). $\phi(\nu)$ follows immediately from l_0, and λ_1 and λ_2 were computed from l_1 and l_2 by the use of (74). Λ_1 and Λ_2 then follow with the aid of (83) and (123) respectively.

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Quantities in expression for lift force

ν	0	0·05	0·1	0·5	1·0	1·5
$\phi(\nu)$ (radians)	0	0·2902	0·4079	0·7264	0·8669	0·9430
$\Lambda_1(\nu)$	0	0·1328	0·1882	0·3929	0·5214	0·6073
$\Lambda_2(\nu)$	0	−0·1033	−0·1411	−0·2758	−0·3597	−0·4178

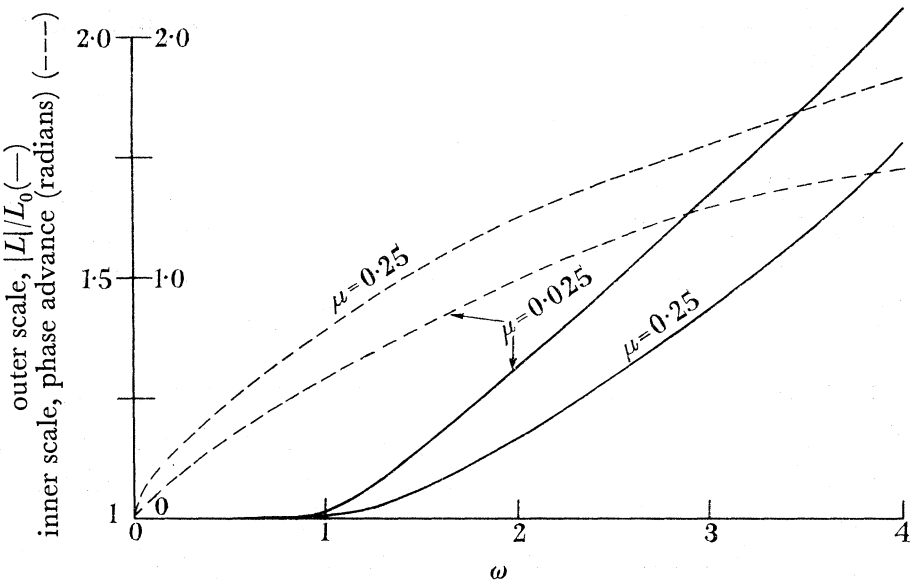

Figure 8. Oscillating deflexion angle: lift amplitude (solid lines) and phase advance (broken lines) for two values of μ.

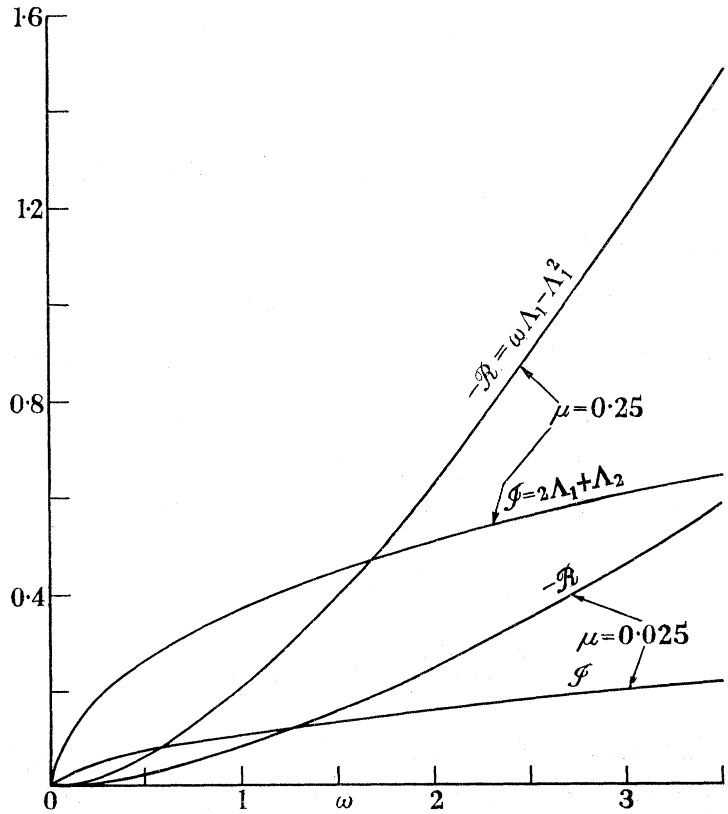

Figure 9. Plunging motion: additional lift contributions due to jet for two values of μ (see equation (124)).

(The limiting behaviour as $\nu \to 0$ is $\phi(\nu) \doteqdot 2\Lambda_1(\nu) \doteqdot -2\Lambda_2(\nu) \doteqdot \nu^{\frac{1}{2}}$, the terms but for which these equalities would be exact being of order $\nu \ln \nu$.)

As an example the real and imaginary contributions to the non-dimensional lift which depend on the strength of the jet, namely the terms $\Lambda_1^2 - \omega\Lambda_1$ and $2\Lambda_1 + \Lambda_2$, are plotted in figure 9 as functions of ω for the same two values of μ as used previously.

The basic element $\mathrm{i}\omega - \frac{3}{4}\omega^2$ in (124) is not however the same as in flow in the absence of a jet, when the classical value

$$L = \pi\rho U^2 \beta c \, \mathrm{i}\omega \left[\tfrac{1}{4}\mathrm{i}\omega + \frac{K_1(\frac{1}{2}\mathrm{i}\omega)}{K_0(\frac{1}{2}\mathrm{i}\omega) + K_1(\frac{1}{2}\mathrm{i}\omega)}\right],$$

involving the so-called Theodorsen function holds (see, for instance, von Kármán & Sears 1938). The essential difference is that in the absence of a jet the circulation $\Gamma_\infty(t)$ is zero by Kelvin's theorem, but a jet extending to infinity, however weak, alters the connectivity of the region of potential flow and the theorem no longer applies.

7. Pitching motion about the trailing edge, with jet tangential

We now consider the case indicated in figure 1 (*c*) in which the wing executes oscillations in pitch about the trailing edge $x = c$ and the jet emerges at all times tangential to the chord line. The solution for pitching motion about a point other than the trailing edge could then be obtained by superposition of an appropriate multiple of the plunging solution of § 6. Accordingly, we suppose the wing ordinate is given by

$$h_W = \alpha(x-c)\,\mathrm{e}^{\mathrm{i}\omega Ut/c},$$

in which case

$$\left.\begin{aligned} h_W &= \alpha(x-c)\,\mathrm{e}^{\mathrm{i}\omega Ut/c}, \\ w_W &= U\alpha\left[1 + \mathrm{i}\omega\left(\frac{x-c}{c}\right)\right]\mathrm{e}^{\mathrm{i}\omega Ut/c} \end{aligned}\right\} \tag{125}$$

and the right-hand side of (3) becomes

$$U\alpha\left\{\left[1+\mathrm{i}\omega\left(\frac{x-c}{c}\right)\right]\left[1-\left(\frac{x-c}{x}\right)^{\frac{1}{2}}\right] - \tfrac{1}{2}\mathrm{i}\omega\left(\frac{x-c}{x}\right)^{\frac{1}{2}}\right\}, \tag{126}$$

while equations (4) and (6) remain as before.

7·1. *Governing equations*

An appropriate transformation for this case, corresponding to equations (111) of the previous section, is

$$\left.\begin{aligned} h_J &= \alpha\mu c[\bar{x} + (\mu c/x)^{\frac{1}{2}} h_2(\bar{x})]\,\mathrm{e}^{\mathrm{i}\omega Ut/c}, \\ \gamma_J &= 2U\alpha(\mu c/x)^{\frac{1}{2}} g_2(\bar{x}), \end{aligned}\right\} \tag{127}$$

which leads on substitution in (3), (4) and (6) to the equations

$$\left(\mathrm{i}\nu + \frac{\mathrm{d}}{\mathrm{d}\bar{x}}\right) h_2(\bar{x}) + \frac{1}{\pi}\int_0^\infty \left(\frac{\bar{x}}{\bar{x}_1}\right)^{\frac{1}{2}} \frac{g_2(\bar{x}_1)\,\mathrm{d}\bar{x}_1}{\bar{x}_1 - \bar{x}} = -(1+\tfrac{1}{2}\mathrm{i}\omega)\,\bar{x}^{\frac{1}{2}} - \mathrm{i}\nu\bar{x}^{\frac{3}{2}}, \tag{128}$$

$$\left(\mathrm{i}\nu + \frac{\mathrm{d}}{\mathrm{d}\bar{x}}\right) g_2(\bar{x}) = -h_2'''(\bar{x}). \tag{129}$$

These may be combined as were (112) and (113) to yield the single equation

$$\begin{aligned} \mathscr{L}h_2(x) &= [\mathrm{i}\nu A_2 - \tfrac{1}{2}(1+\mathrm{i}\omega)]\,x^{-\frac{1}{2}} - \mathrm{i}\nu(\tfrac{5}{2}+\tfrac{1}{2}\mathrm{i}\omega)\,x^{\frac{1}{2}} + \nu^2 x^{\frac{3}{2}} \\ &\equiv \mathrm{i}\nu(ax^{-\frac{1}{2}} + bx^{\frac{1}{2}} - \mathrm{i}\nu x^{\frac{3}{2}}), \end{aligned} \tag{130}$$

say, where
$$A_2 = \frac{1}{\pi}\int_0^\infty x^{-\frac{1}{2}} g_2(x)\,\mathrm{d}x, \tag{131}$$

$\mathscr{L}$ is the operator previously defined (by equation (22)), and the bar over x has been omitted. In this case the simplest course is to solve directly by the method of § 3. Laplace transformation gives

$$\left(1+\frac{\mathrm{i}\nu}{\xi}\right)^2 \bar{h}_2(\xi) + \frac{1}{\pi}\int_0^\infty \left(\frac{\xi_1}{\xi}\right)^{\frac{1}{2}} \frac{\xi_1 \bar{h}(\xi_1)\,\mathrm{d}\xi_1}{\xi_1-\xi} = \mathrm{i}\nu\sqrt{\pi}\left(\frac{a}{\xi^{\frac{3}{2}}}+\frac{b}{2\xi^{\frac{5}{2}}}-\frac{3\mathrm{i}\nu}{4\xi^{\frac{7}{2}}}\right), \tag{132}$$

with solution

$$\bar{h}_2(\xi) = \frac{\mathrm{i}\nu\sqrt{\pi}\,\mathrm{e}^{-k_0}}{\xi\sqrt{\{\xi^6+(\xi+\mathrm{i}\nu)^4\}}}\left[a\xi^2+\tfrac{1}{2}b\xi(1-k_1\xi)-\tfrac{3}{4}\mathrm{i}\nu\{1-k_1\xi-\tfrac{1}{2}(k_2-k_1^2)\,\xi^2\}\right]\mathrm{e}^{\Omega(\xi)}, \tag{133}$$

where k_0, k_1 and k_2 are the coefficients in the expansion of $X(\zeta)$ defined by (48), and are related to l_0, l_1 and l_2 by (80). (If $\nu > 2$ the last formulae would also contain contributions from β_2 and β_3, but as before no solution exists in this case.)

As before, $\bar{h}_2(\xi)$ must now be continued to the whole ζ-plane, with the aid of equation (60). For $\nu < 2$ this produces the factors $(\zeta-\alpha_2)(\zeta-\alpha_3)$ in the numerator, the second of which must be cancelled by choosing A_2 appropriately, as otherwise there would be a pole at $\zeta = \alpha_3$ which lies in the right-hand half plane. We must therefore choose A_2 in such a way that the expression in square brackets in (133) vanishes at $\xi = \alpha_3$; this requires

$$A_2 = \frac{3}{4}\left(\frac{\mathrm{i}\nu}{\alpha_3}+1\right)\left(\frac{1}{\alpha_3}-k_1\right)+\tfrac{1}{2}(1+\tfrac{1}{2}\mathrm{i}\omega)\left(\frac{1}{\mathrm{i}\nu}+\frac{1}{\alpha_3}-k_1\right)-\tfrac{3}{8}\mathrm{i}\nu(k_2-k_1^2)$$
$$(\sim -\tfrac{1}{4}(1-\mathrm{i}\omega)\,\nu^{-\frac{1}{2}} \quad \text{as} \quad \nu\to 0), \tag{134}$$

and when A_2 has this value the stated expression equals

$$\frac{1}{\alpha_3}(\xi-\alpha_3)\left(c\xi+\frac{3\mathrm{i}\nu}{4}\right), \tag{135}$$

where
$$c = \frac{3\mathrm{i}\nu}{4}\left(-k_1+\frac{1}{\alpha_3}\right)+(\tfrac{5}{4}+\tfrac{1}{4}\mathrm{i}\omega)$$
$$(=\tfrac{1}{2}(1+\tfrac{1}{2}\mathrm{i}\omega)+O(\nu^{\frac{1}{2}}) \text{ for small } \nu).$$

Then since $\mathrm{i}\nu\sqrt{\pi}\,\mathrm{e}^{-k_0} = -\alpha_3/A_0$, by (65), we finally obtain

$$\bar{h}_2(\zeta) = -(c\zeta+\tfrac{3}{4}\mathrm{i}\nu)\,\mathrm{e}^{-Z(\zeta)}/A_0\zeta^2(\zeta-\alpha_2). \tag{136}$$

This can be expressed in terms of the solution for the plunging case of § 6: on forming the Laplace transform of (117) we have

$$\bar{h}_1(\zeta) = -\bar{h}_0(\zeta)/2A_0\zeta, \tag{137}$$

whence
$$\bar{h}_2(\zeta) = \left(2c+\frac{3\mathrm{i}\nu}{2\zeta}\right)\bar{h}_1(\zeta) \quad \text{and} \quad h_2(x) = \frac{3\mathrm{i}\nu}{2}\int_0^x h_1(x)\,\mathrm{d}x+2ch_1(x). \tag{138}$$

7·2. *Lift force in pitching motion*

Substitution of (125) and (127) in (9) in this case gives

$$\Gamma_\infty = \pi U\alpha c(1-\tfrac{1}{4}\mathrm{i}\omega+2\mu A_2)\,\mathrm{e}^{\mathrm{i}\omega Ut/c},$$

and the expression (8) for the lift force now becomes

$$\frac{L}{\pi\rho U^2 \alpha c\, e^{i\omega Ut/c}} = 1 + \tfrac{1}{2}i\omega + \tfrac{1}{4}\omega^2 + 2\mu[(1+\tfrac{1}{2}i\omega)A_2 - i\nu B_2], \tag{139}$$

where A_2 is given by (134) and $B_2 = \frac{1}{\pi}\int_0^\infty x^{\frac{1}{2}} g_2(x)\,dx.$

Here again, the limit as $\mu \to 0$ differs from the classical value involving the Theodorsen function, for the same reasons as indicated in § 6·3. The calculation of B_2 presents formidable algebraic complications that have not been attempted but it seems most likely that νB_2, like A_2, behaves like $\nu^{-\frac{1}{2}}$ as $\nu \to 0$, and on physical grounds one would expect the singular behaviour contributed by the two terms in this limit to cancel out.

8. Concluding remarks

The finding that solutions to the problem for real frequencies ω exist only when $\nu = \mu\omega$ lies below a certain critical value $\nu_{\text{crit.}}$ say (which is 2 in the limit $\mu \to 0$, but is given by the more complicated expression (101) for larger values of μ), was unexpected, and a completely satisfactory explanation has not been found, although two possibilities suggest themselves. (i) It would not be altogether surprising if linearized theory, which supposes velocities normal to the boundary to be small compared with that of the undisturbed stream, broke down at high frequencies; moreover, the boundary condition applied at the jet, equation (5), would also be suspect physically if the velocity of the jet normal to its boundary were comparable to that of the flow within it. However, the fact that linearized theory becomes suspect at high frequencies does not necessarily mean that solutions should cease to exist for the linearized model. (ii) A more likely explanation to the author's mind is that $\nu_{\text{crit.}}$ represents a dynamical stability boundary beyond which steady oscillations of the system are impossible. It is possible to obtain solutions for oscillations in which the real part of ν is greater than $\nu_{\text{crit.}}$, but ν must then be complex with a negative imaginary part, so such an oscillation would be divergent.†

A stability boundary of this kind does not seem to have been predicted before, possibly because comparable flows involving wakes and regions of separation are difficult to study analytically, but it may well occur, and presumably in the latter cases the breakdown would lead to turbulence. In this connexion it may also be noted that ν can be looked on as a Strouhal number for the jet, since if δ is the jet width and V the velocity within it (which have up to now been thought of as zero and infinity respectively) we should have $J = \rho V^2\delta$, and $\mu = \frac{1}{2}(V/U)^2\,\delta/c$, by (7), and on setting $\omega U/c = 2\pi f$, the analogue of the wake Strouhal number defined by Roshko (1961) would be

$$S^* = \frac{f\delta}{V} = \frac{\nu}{\pi}\left(\frac{U}{V}\right)^3.$$

† The condition for a solution to exist is that $\mathscr{I}\beta_1 < 0$. If we write $\nu = n + ip$, this condition becomes

$$p < -n^{\frac{2}{3}}(n^{\frac{1}{3}} - 2^{\frac{1}{3}}),$$

so when $n = \mathscr{R}\nu > 2$, $p = \mathscr{I}\nu$ must be negative if a solution is to exist. The growth in amplitude per cycle for a possible motion cannot therefore be less than $\exp 2\pi\left[1 - \left(\frac{2}{n}\right)^{\frac{1}{3}}\right]$.

Such a number might well characterize the breakdown of an oscillatory flow. (But when the present solution is repeated for a negative value of μ, to represent a wake, no cutoff frequency is found, although the inviscid flow becomes indeterminate above $|\nu| = 2$.)

Comment is also necessary on the breakdown of the solution for transient motions proposed in Spence (1961 *b*). This solution was obtained for small times after the deflexion of a jet flap by omitting $\partial h/\partial x$ in comparison with $\partial h/\partial t$—or in the context of the present paper, for large values of ν by omitting $\mathrm{d}h/\mathrm{d}\bar{x}$ in comparison with $i\nu h$, and $\mathrm{d}g/\mathrm{d}\bar{x}$ in comparison with $i\nu g$, in equations (24) and (25). The latter then admit a similarity solution in terms of the variable $\nu^{\frac{2}{3}}x$, and the corresponding equations for time-dependent motion admit one which depends only on $x/t^{\frac{2}{3}}$. It is clear, however, that the full equations (24) and (25) do not have a solution as $\nu \to \infty$, so this approach must be discarded. The precise reason for the breakdown of what appeared a plausible approximation is that $g(\bar{x})$ behaves like $\ln \bar{x}$ near $\bar{x} = 0$, so $i\nu g$ is dominated in this neighbourhood by the term discarded, however large ν becomes. Erickson (1962) has proposed a solution of the similarity form for large ν slightly different from that of Spence (1961 *b*), but this suffers from the same objection. It is clear that the approach also fails for transient motions, since these are obtained by Fourier synthesis of solutions for all frequencies, but those for frequencies $\omega > \nu_{\text{crit.}}/\mu$ do not exist.

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Appendix A. Derivation of equation (16)

Equation (14) can be rearranged as

$$-2\pi U\left(\frac{\mathrm{i}\omega}{c}+\frac{\mathrm{d}}{\mathrm{d}x}\right)h_J(x)=\int_c^\infty\left[\frac{x_1(x_1-c)}{x(x-c)}\right]^{\frac{1}{2}}\frac{\gamma_J(x_1)\,\mathrm{d}x_1}{x_1-x}-\frac{\Gamma}{x^{\frac{1}{2}}(x-c)^{\frac{1}{2}}}\tag{A 1}$$

where

$$\Gamma=\int_c^\infty\left(\frac{x_1}{x_1-c}\right)^{\frac{1}{2}}\gamma_J(x_1)\,\mathrm{d}x_1.$$

The first term on the right, with the substitution $x_1 = xu$, is

$$\int_{c/x}^\infty\left(\frac{xu-c}{x-c}\right)^{\frac{1}{2}}\frac{u^{\frac{1}{2}}\gamma_J(xu)\,\mathrm{d}u}{u-1}.$$

Its derivative with respect to x is

$$\int_{c/x}^\infty\left(\frac{xu-c}{x-c}\right)^{\frac{1}{2}}\frac{u^{\frac{3}{2}}\gamma_J'(xu)\,\mathrm{d}u}{u-1}-\tfrac{1}{2}c\int_{c/x}^\infty\frac{u^{\frac{1}{2}}\gamma_J(xu)\,\mathrm{d}u}{(x-c)^{\frac{1}{2}}(xu-c)^{\frac{1}{2}}}$$

$$=\int_c^\infty\left[\frac{x_1(x_1-c)}{x(x-c)}\right]^{\frac{1}{2}}\left(\frac{1}{x_1-x}+\frac{1}{x}\right)\gamma_J'(x_1)\,\mathrm{d}x_1-\frac{\frac{1}{2}c\Gamma}{x^{\frac{3}{2}}(x-c)^{\frac{1}{2}}}\tag{A 2}$$

(the term arising from differentiation of the lower limit of integration is absent since

$$\lim_{u\to c/x}(xu-c)^{\frac{1}{2}}\gamma_J(xu)=0),$$

and that of the second is

$$\frac{x-\frac{1}{2}c}{x^{\frac{3}{2}}(x-c)^{\frac{3}{2}}}\Gamma.\tag{A 3}$$

The sum of (A 2) and (A 3) gives

$$-2\pi U\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{i}\omega}{c}+\frac{\mathrm{d}}{\mathrm{d}x}\right)h_J(x)=\int_c^\infty\left[\frac{x_1(x_1-c)}{x(x-c)}\right]^{\frac{1}{2}}\frac{\gamma_J'(x_1)\,\mathrm{d}x_1}{x_1-x}+\frac{\frac{1}{2}c}{x^{\frac{3}{2}}(x-c)^{\frac{1}{2}}}\int_c^\infty\frac{\gamma_J(x_1)\,\mathrm{d}x_1}{x_1^{\frac{1}{2}}(x_1-c)^{\frac{1}{2}}}.\tag{A 4}$$

Adding (A 4) to $\mathrm{i}\omega/c$ times (A 1) we obtain

$$-2\pi U\left(\frac{\mathrm{i}\omega}{c}+\frac{\mathrm{d}}{\mathrm{d}x}\right)^2h_J(x)=\int_c^\infty\left[\frac{x_1(x_1-c)}{x(x-c)}\right]^{\frac{1}{2}}\frac{(\mathrm{i}\omega/c+\mathrm{d}/\mathrm{d}x_1)\,\gamma_J(x_1)\,\mathrm{d}x_1}{x_1-x}$$

$$-\frac{1}{x^{\frac{1}{2}}(x-c)^{\frac{1}{2}}}\int_c^\infty\left(\frac{x_1}{x_1-c}\right)^{\frac{1}{2}}\left(\frac{\mathrm{i}\omega}{c}-\frac{c}{2xx_1}\right)\gamma_J(x_1)\,\mathrm{d}x_1.\tag{A 5}$$

On substitution for $[(\mathrm{i}\omega/c)+(\mathrm{d}/\mathrm{d}x)]\gamma_J$ from (15) this yields (16).

Appendix B. Roots of cubics

$\alpha_1, \alpha_2, \alpha_3$ are defined by (54) as the roots of

$$\xi^3-i(\xi+\mathrm{i}\nu)^2=0.\tag{B 1}$$

Writing $\xi=\mathrm{i}x$, we see that $\alpha_j=\mathrm{i}x_j$, where $x_j(\nu)$ are the roots of

$$p(x,\nu)\equiv x^3-(x+\nu)^2=0.\tag{B 2}$$

Likewise $\beta_1, \beta_2, \beta_3$ are the roots of

$$\xi^3+\mathrm{i}(\xi+\mathrm{i}\nu)^2=0,\tag{B 3}$$

and writing $\xi=-\mathrm{i}x$ here, we see that $\beta_j(\nu)=-\mathrm{i}x_j(-\nu)$. We therefore examine the roots of (B 2) for positive and negative values of ν.

The turning points of $p(x)$ are where $p'(x) = 0$, namely $x = \frac{1}{3}(1 \pm k)$, where $k = \surd(1+6\nu)$, and simple algebra shows that

$$p(\tfrac{1}{3}+\tfrac{1}{3}k)\,p(\tfrac{1}{3}-\tfrac{1}{3}k) = \nu^3(\nu+\tfrac{4}{27}). \tag{B 4}$$

There are three real roots if this product is negative, i.e. if $-\frac{4}{27} < \nu < 0$. For all other values of ν one root is real and two are complex conjugates.

Since $p(1) = -\nu(\nu+2)$, which is negative if $\nu > 0$ or $\nu < -2$, the real root is greater than unity for these values of ν, and since

$$x_1 + x_2 + x_3 = 1 \tag{B 5}$$

it follows that the (equal) real parts of the complex roots are then negative.

Applying this result for $\nu > 0$ we see that the roots of (B 1) are of the form

$$\alpha_1 = \mathrm{i}(1+2b), \quad \left.\begin{matrix}\alpha_2\\ \alpha_3\end{matrix}\right\} = \mp a - \mathrm{i}b, \tag{B 6}$$

with $a, b > 0$ (and $a = b = 0$ when $\nu = 0$). These are distributed in the manner described in § 3·3.

The sums of the roots singly and in pairs, and their product, are

$$\left.\begin{aligned} &\Sigma\alpha_i = \mathrm{i}, \quad \sum_{(i\neq j)} \alpha_i\alpha_j = 2\nu, \quad \alpha_1\alpha_2\alpha_3 = -\mathrm{i}\nu^2 \\ \text{and} \quad &\Sigma\frac{1}{\alpha_i} = \frac{2\mathrm{i}}{\nu}, \quad \Sigma\frac{1}{\alpha_i^2} = -\frac{2}{\nu^2}. \end{aligned}\right\} \tag{B 7}$$

Again, if $-\frac{4}{27} < \nu < 0$, since $p(0) < 0 < p(1)$, and since the turning points of $p(x)$ both lie between 0 and 1, all three roots lie within this interval; therefore for $0 < \nu < \frac{4}{27}$ the roots of (B 3) are pure imaginary, namely

$$\beta_j = -\mathrm{i}x_j \ (j = 1, 2, 3) \quad 0 < x_j < 1, \tag{B 8}$$

where the x_j satisfy (B 5).

When $$\nu = -\tfrac{4}{27}, \quad x_1 = x_2 = \tfrac{4}{9}, \quad x_3 = \tfrac{1}{9}.$$

When $-2 < \nu < -\frac{4}{27}$, x_1 and x_2 become complex conjugates, and $0 < x_3 < 1$. Thus the root of (B 3) for $\frac{4}{27} < \nu < 2$ are of the form

$$\left.\begin{matrix}\beta_1\\ \beta_2\end{matrix}\right\} = \pm c - \mathrm{i}d, \quad \beta_3 = -i(1-2d), \tag{B 9}$$

with $c > 0$, $\frac{1}{2} > d > 0$.

However for $\nu < -2$, $x_3 > 1$ and the roots are then as above but with $d < 0$, i.e. β_1 and β_2 are then in the upper-half plane for $\nu > 2$. The critical case $\nu = 2$ corresponds to $c = 2$, $d = 0$.

The sums corresponding to (B 7) are

$$\left.\begin{aligned} &\Sigma\beta_i = -\mathrm{i}, \quad \sum_{i\neq j} \beta_i\beta_j = -2\nu, \quad \beta_1\beta_2\beta_3 = \mathrm{i}\nu^2 \\ \text{and} \quad &\Sigma\frac{1}{\beta_1} = \frac{2\mathrm{i}}{\nu}, \quad \Sigma\frac{1}{\beta_i^2} = -\frac{2}{\nu^2}. \end{aligned}\right\} \tag{B 10}$$

We note that $$\Sigma\left(\frac{1}{\alpha_i} - \frac{1}{\beta_i}\right) = \Sigma\left(\frac{1}{\alpha_i^2} - \frac{1}{\beta_i^2}\right) = 0. \tag{B 11}$$

Appendix C. Values of $\mathscr{R}\, l_0$, $\mathscr{I}\, l_1$ and $\mathscr{R}\, l_2$

For the purpose of this calculation it is more convenient to write the roots α_i in the form

$$\alpha_1 = \mathrm{i}\,|\alpha_1|, \quad \alpha_2 = r\,\mathrm{e}^{\mathrm{i}(\theta-\pi)}, \quad \alpha_3 = r\,\mathrm{e}^{-\mathrm{i}\theta} \quad (0 < \theta < \tfrac{1}{2}\pi), \tag{C 1}$$

instead of (B 6). The modulus of their product is then

$$r^2\,|\alpha_1| = \nu^2. \tag{C 2}$$

Likewise the roots β_i for $2 > \nu > \frac{4}{27}$ can be written

$$\beta_3 = -\mathrm{i}\,|\beta_3|, \quad \beta_1 = \rho\,\mathrm{e}^{-\mathrm{i}\theta}, \quad \beta_2 = \rho\,\mathrm{e}^{\mathrm{i}(\phi-\pi)} \quad (0 < \phi < \tfrac{1}{2}\pi) \tag{C 3}$$

and

$$\rho^2\,|\beta_3| = \nu^2. \tag{C 4}$$

Substitution of (C 1) and (C 3) in the first expression of (77) gives

$$\begin{aligned} \mathscr{R}\, l_0 &= \tfrac{1}{4}[\ln|\alpha_1| - 2\ln r + \ln|\beta_3| + 2\ln\rho] \\ &= \tfrac{1}{2}\ln|\alpha_1| \end{aligned} \tag{C 5}$$

by (C 2) and (C 4). When $0 < \nu < \frac{4}{27}$ we write instead $\beta_i = -\mathrm{i}\,|\beta_i|$ where $|\beta_1\beta_2\beta_3| = \nu^2$, and easily recover the same result.

The results for $\mathscr{I}\, l_1$ and $\mathscr{R}\, l_2$ are also obtained by substitution in the appropriate equation of (77). We do this here only using the forms for β_i that apply for $2 > \nu > \frac{4}{27}$, but it is easy to verify that the results are unaffected by the change in form of β_1 and β_2 when $0 < \nu < \frac{4}{27}$.

First, we find

$$\mathscr{I}\, l_1 = \frac{1}{2}\left[\left(\frac{1}{2\,|\alpha_1|} + \frac{\sin\theta}{r}\right) - \left(\frac{1}{2\,|\beta_3|} + \frac{\sin\phi}{\rho}\right)\right]. \tag{C 6}$$

To simplify this expression we note that the sum

$$\alpha_2\alpha_3 + \alpha_1(\alpha_2+\alpha_3) = -r^2 + 2r\,|\alpha_1|\sin\theta = 2\nu$$

by (B 7). Division through by $2r^2\,|\alpha_1| = 2\nu^2$ gives

$$-\frac{1}{2\,|\alpha_1|} + \frac{\sin\theta}{r} = \frac{1}{\nu}. \tag{C 7}$$

Likewise

$$\frac{1}{2\,|\beta_3|} + \frac{\sin\phi}{\rho} = \frac{1}{\nu}. \tag{C 8}$$

Substitution of the last two results in (C 6) gives

$$\mathscr{I}\, l_1 = \frac{1}{2\,|\alpha_1|} = \frac{\sin\theta}{r} - \frac{1}{\nu} = \mathscr{I}\,\frac{1}{\alpha_2} - \frac{1}{\nu}. \tag{C 9}$$

Again,

$$\begin{aligned} \mathscr{R}\, l_2 &= \frac{1}{2}\left[\left(\frac{1}{2\,|\alpha_1|^2} + \frac{\cos 2\theta}{r^2}\right) - \left(-\frac{1}{2\,|\beta_3|^2} + \frac{\cos 2\phi}{\rho^2}\right)\right] \\ &= \frac{1}{4}\left[\left(-\frac{1}{a_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}\right) - \left(\frac{1}{\beta_3^2} + \frac{1}{\beta_1^2} + \frac{1}{\beta_2^2}\right)\right]. \end{aligned} \tag{C 10}$$

Use of (B 11) reduces this to

$$\mathscr{R}\, l_2 = -\frac{1}{2\alpha_1^2} = \tfrac{1}{2}\,|\alpha_1|^{-2}. \tag{C 11}$$

Appendix D. Behaviour of $\Theta(\xi)$ when ν is small

Integrating (67) by parts, we can write

$$\Theta(\xi) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\theta(\xi_1)-\pi}{\xi_1-\xi}\,d\xi_1 = -\tfrac{1}{2}\ln|\xi| + l(\xi), \tag{D 1}$$

where

$$l(\xi) = -\frac{1}{\pi}\int_{-\infty}^{0}\theta'(\xi_1)\ln|\xi_1-\xi|\,d\xi_1 = \frac{1}{2\pi i}\int_{-\infty}^{0}\sum_{i=1}^{3}\left(\frac{1}{\xi_1-\alpha_i}-\frac{1}{\xi_1-\beta_i}\right)\ln|\xi_1-\xi|\,d\xi_1. \tag{D 2}$$

Now using the expansions for α_1, β_1 quoted in equation (62) we have

$$\frac{1}{2i}\left(\frac{1}{\xi_1-\alpha_1}-\frac{1}{\xi_1-\beta_1}\right) = \frac{1}{\xi_1^2+1}\left[1+\frac{4i\nu\xi_1}{\xi_1^2+1}+O(\nu^2)\right]. \tag{D 3}$$

The contribution this term makes to $l(\xi)$ is found on integration to be

$$\frac{1}{\pi}\int_{-\infty}^{0}\frac{\ln|\xi_1-\xi|\,d\xi_1}{\xi_1^2+1} - \frac{2i\nu}{\pi}\ln|\xi| + \frac{2i\nu}{\pi}\left(\frac{\ln|\xi|-\frac{1}{2}\pi\xi}{\xi^2+1}\right)+O(\nu^2). \tag{D 4}$$

The first term on the left here can be expressed in terms of $\Theta_0(\xi)$, the value of $\Theta(\xi)$ when $\nu = 0$, since

$$\Theta_0(\xi) = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\cot^{-1}|\xi_1|\,d\xi_1}{\xi_1-\xi} = -\tfrac{1}{2}\ln|\xi| + \frac{1}{\pi}\int_{-\infty}^{0}\frac{\ln|\xi_1-\xi|\,d\xi_1}{\xi_1^2+1}. \tag{D 5}$$

It remains to calculate the contributions to $l(\xi)$ from the α_2, α_3 and β_2, β_3 terms. With use of the expansions (62), this is found to be

$$\frac{1}{\pi i}\int_{-\infty}^{0}\left(\frac{[\xi_1+i\nu(1-\frac{3}{2}\nu)]}{[\;\;]^2-\nu^3}-\frac{\{\xi_1+i\nu(1+\frac{3}{2}\nu)\}}{\{\;\;\}^2+\nu^3}\right)\ln|\xi_1-\xi|\,d\xi_1$$

$$= \frac{\nu}{\pi i}\int_{-\infty}^{0}\left\{\frac{3i}{(\eta_1+i)^2}+\frac{2}{(\eta_1+i)^3}+O(\nu)\right\}(\ln\nu+\ln|\eta_1-\eta|)\,d\eta_1 \tag{D 6}$$

where $\eta = \xi/\nu$.

On integration by parts this can be written

$$-\frac{2\nu}{\pi i}\ln|\xi| + \frac{\nu}{\pi i}\Psi(\eta) + O(\nu^2\ln\nu) \tag{D 7}$$

say, where

$$\Psi(\eta) = \frac{3i\pi-2}{(\eta+i)^2}(\ln|\eta|+\tfrac{1}{2}i\pi) - \frac{i}{\eta+i} \tag{D 8}$$

$l(\xi)$ is the sum of (D 4) and (D 7), and substitution in (D 1) with use of (D 5) yields equation (85) of the main text.

The coefficients in the expansion of l for small ξ defined by (73) are found by direct evaluation of the respective integrals, using similar approximations to those above; the results are

$$l_0 = \left(\frac{2i}{\pi}\right)\nu\ln\nu + \left(1+\frac{i}{\pi}\right)\nu, \quad l_1 = -\frac{1}{\pi}\ln\nu + \frac{i}{2}, \quad l_2 = -\frac{2i}{\pi\nu} + \frac{1}{2}. \tag{D 9}$$

(The first two of these can also be found by letting $\xi \to 0$ in $l(\xi)$ and $l'(\xi)$ respectively, but a further term in the expansion for small ν would be necessary to get the third in this way.)